"From Problems to Numbers and Back"
Lecture Notes to 'A Discipline-neutral Introduction to Mathematical Modeling'

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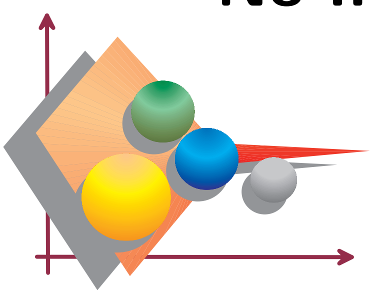
Chapter 1

No model without a Purpose



'PERHAPS REALITY IS JUST A BETTER MODEL'

T zero minus two days. After months of planning and preparation, the metamorphosis is complete. Acres of wasteland have been magically transformed into the venue of the next mega pop festival. Five huge stages have been erected, catering and sanitary facilities, sufficient to serve a thirsty and sweaty crowd, are in place; PR machinery has been running at full steam; nearly one hundred thousand tickets have been sold; and an impressive line up of hip rock stars has been booked. And then ... the e-mail arrives. It does not contain the threat of a terrorist attack, neither the announcement of the main attraction's withdrawal. A short paragraph merely contains the result of a computer predicting air movements in the atmosphere. Something about wind speed, a sharp drop of air pressure... Although the sky still looks perfectly innocent, tables with numbers foretell a 95% chance of a major thunderstorm with dangerously strong gusts of wind to pass over the festival terrain, exactly during the opening night. There is no discussion. The director cancels the festival. Sponsors loose their money, ticket holders receive a refund accompanied by a frustrating little e-mail apology, and two or three management bureaus go bankrupt.

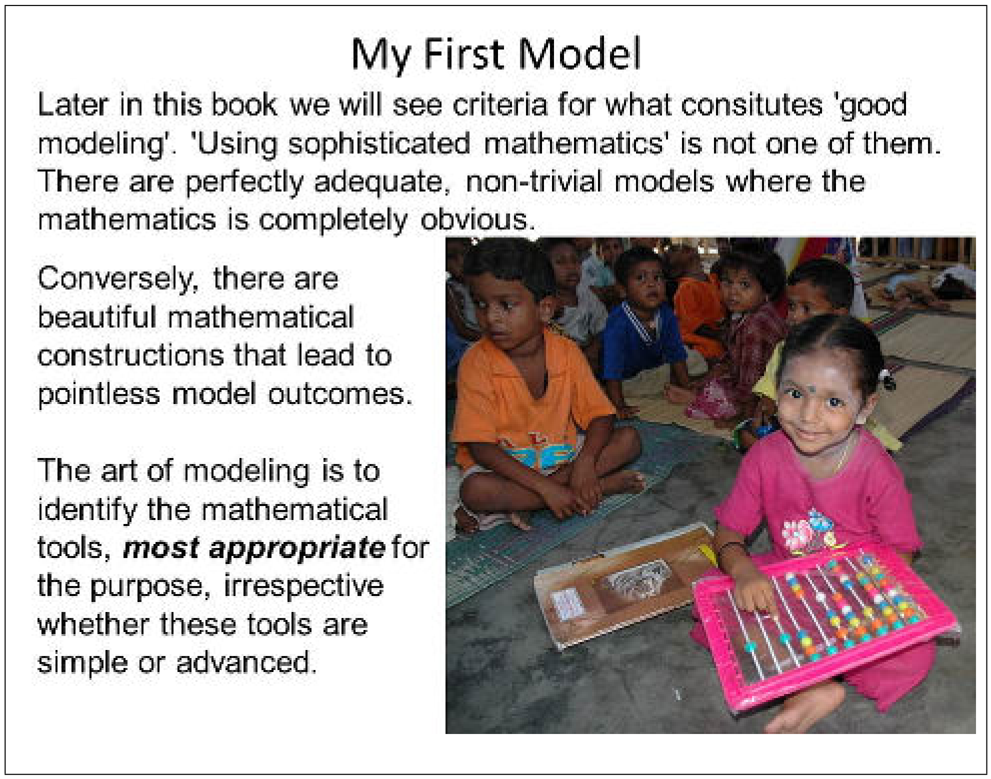
1.1 Models that Everybody Knows

The little story above, although fictitious, gives a hint about the potential impact of using MODELS. Let us analyse what is happening.

People look to the sky to know if it will be raining soon. Dark clouds means: rain expected. The computer program, causing the cancelation of the pop festival in the above story, seems to do something else. It uses the equivalent of 'looking at the sky': it takes recent, accurate measurements of temperatures, humidities and air pressures as input, but it does *not* draw its conclusions on mere inspection of the current state. Instead, it runs calculations in an attempt to predict the future.

Running calculations to predict the future in itself is not magic. Suppose you want to know if you can afford to buy, say a 25 Euro book, and have enough money left to buy food later today. You do a number of things:

- *Collect data* : do a 'measurement' to assess the amount of money in your wallet. This measurement could be more or less accurate (do you also count the copper coins?); sometimes you know that a measurement can have a systematic error (if you ignore the copper coins, only counting bank notes and silver coins, you are certain that your measurement is an underestimate);
- *Find out the value of one or more known constants* : in this case, the price of the book: 25 Euro; a meal: at least 5 Euro;
- *Perform a mathematical operation* [1]: in this case, you subtract the known, exact value (25 Euro) from the 'measured' value (the amount of money in your wallet); the result is some number - typically with some uncertainty (in this case, the uncertainty comes from ignoring the copper coins);
- *Interpret the outcome of this* MATHEMATICAL OPERATION (=subtraction, producing a number) in terms of your initial problem (='will I have enough money left for buying food'). This interpretation yields an answer such as 'yes', 'no', 'probably', 'unlikely' - or it can even prompt you to do a new calculation or a new measurement.



All the above steps are trivial - in fact, few people will call this 'applying a MODEL'. It is immediately clear how the future, regarding your options for buying food, will look like. 'Predicting the future' can be as simple as merely doing a single subtraction. It can also be as complicated as running a weather simulator. In both cases, however, we perform the same kinds of steps.

Some models are set up to predict future events, such as weather models and our simple wallet model. Prediction, however, is by far not the only purpose of models. In the next section we will investigate the various purposes models can have.

[1]The photograph of a little girl doing arithmetic on a toy abacus was taken from http://commons.wikimedia.org/wiki/Abacus#mediaviewer/File:Indian_pre-school_girl_in_pink_shirt_plays_with_abacus.jpg

1.2 Various Kinds of Modeling Purposes

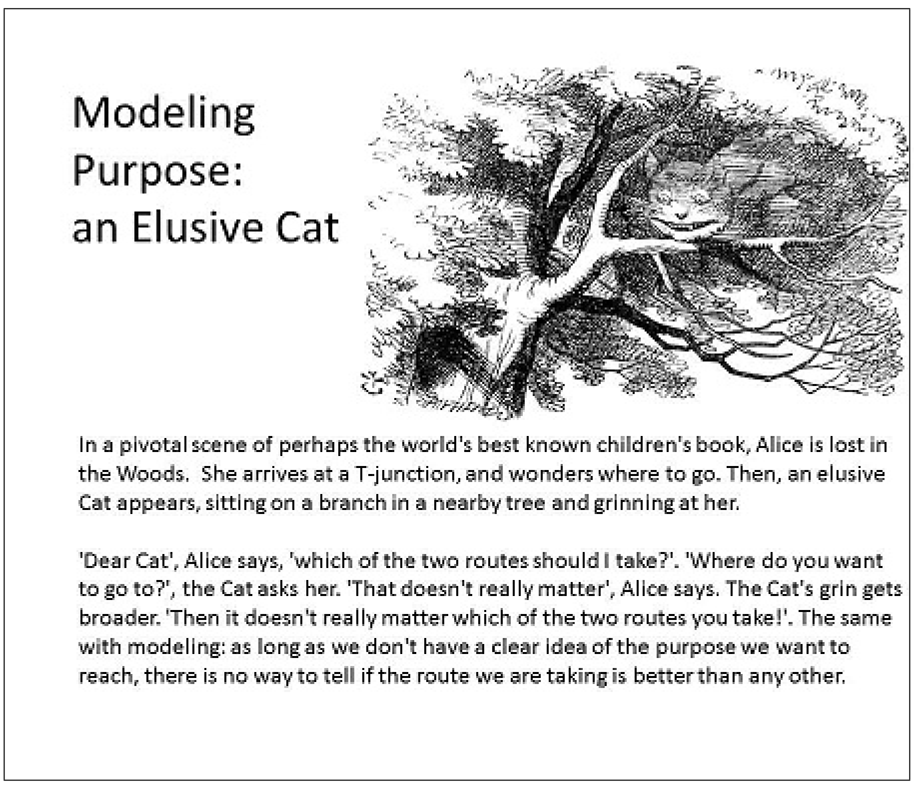
It is important to realize, before developing or using a model what the model should do for you: what is it that the modeler *wants* from the model? This is what we call the MODEL'S PURPOSE. A model, just like medicine, should only be used once the purpose is perfectly clear. Few people would use medicine just to see what will happen: similarly, it is pointless to use a model without clear need. Taking a medicine is a means to an end, namely, to become healthy. Similarly, deploying a model is a means to an end.

For this reason, it is impossible to say that 'a model is good' or 'a model is bad': a model can only be good or bad *with respect to some given purpose*[2].

Purposes can belong to the context of SCIENTIFIC RESEARCH or to the context TECHNOLOGICAL DESIGN.

1.2.1 Purposes of Modeling in the Context of Research

Let us consider the example of our Solar System. We might want to know something about the Solar System: the Solar System could be our object of research. We can study the Solar System in various ways. For instance:



1. *A planet is seen as the visual manifestations of an ancient deity.* This explains why Mars is red (the god of war, associating to fire and blood), why Mercury runs so fast (the messenger of the gods), and Saturn so slowly (slowness being one of the archetypical attributes of the god Saturn).
2. *A planet is seen as a celestial body* , its trajectory therefore is a system of circles centered round the earth, according to Medieval belief. This is the Ptolemaic view, after Ptolmy (approximately AD 90-168) ▷1 . This model predicts the location of planets on the night sky at arbitrary dates to rather good accuracy - sufficient, for instance, to do astrological predictions, or navigation at sea using the stars.
3. *A planet is seen as a point moving round the sun in ellipses.* Johannes Keppler (1571-1630), by observing and analyzing the accurate measurements of Tycho

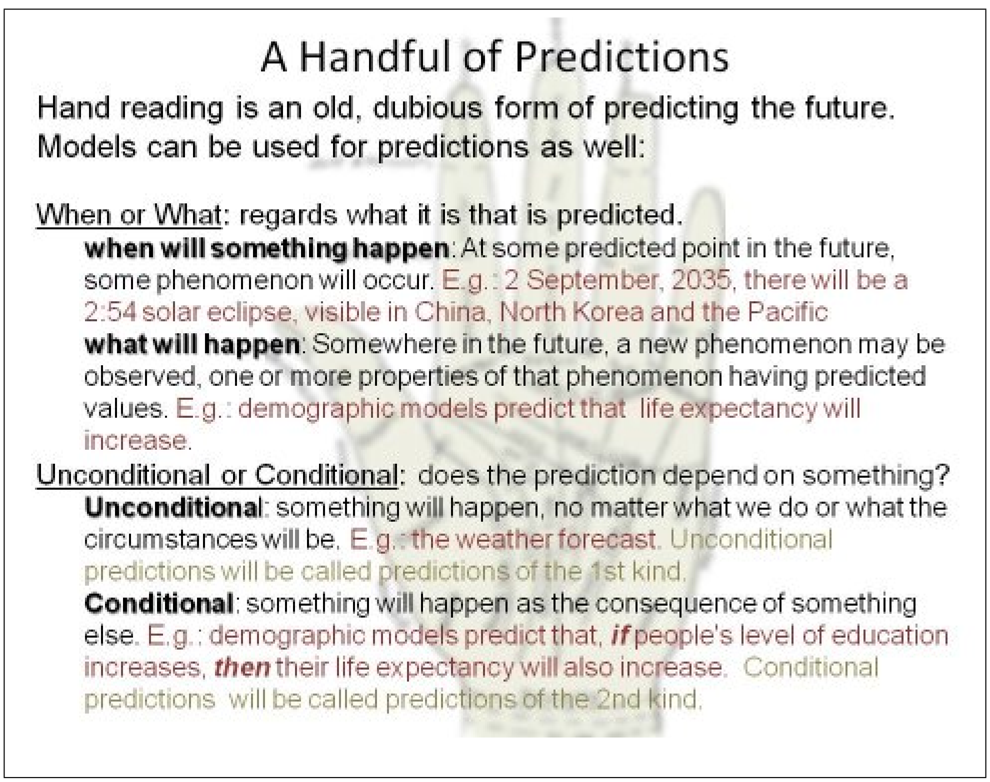
[2] Image 'Alice in Wonderland' from http://commons.wikimedia.org/wiki/John_Tenniel#mediaviewer/File:Alice_par_John_Tenniel_24.png

Brahe (1546-1601), found that there was a relation between the planet's distance to the sun and its speed: the so-called 'equal areas-law' [▷2] . This model was used to summarize the EMPIRICAL findings of earlier astronomers.

4. *A planet is seen as a point mass, moving in the sun's gravity field.* Newton (1642-1727) postulated a model for the interaction between any two bodies with mass. This interaction is a force, experienced by the two bodies. It is the force that pulls the ripe apple from the tree, the unfortunate child from its bicycle, and the cast stone back to the ground. It is called gravity, and Newton postulated that it not only works between earthly objects and our home planet, but also in the cosmos between stars and planets. The merits of Newton's model are, that it explains Keppler's laws, and that it predicts, for instance, whether an approaching meteorite will collide with the earth.

5. *Planets are seen as multiple objects, mutually interacting through gravity.* Newton lacked the mathematical tools to find the orbits of three or more objects (for instance: earth - sun - moon) as a result of their mutual gravity. A general CLOSED-FORM SOLUTION of this so-called three body problem is impossible to find, but high-accuracy numerical approximations are often used instead [▷3] .

Let us investigate, from the above 5 cases, what purposes models for scientific research can have.



- *Ancient deities:* to EXPLAIN X is answering a question of the form 'Why is X the case?'. People have beliefs about a system, e.g., a religious system, containing gods and their attributes. Further, there is a set of observations, *viz.* lights in the night sky at varying locations, having various colors. By assuming correspondences between the two, some 'why' questions could be answered. 'Why is that 'star' red?' Answer: 'because it is Mars, the god of war'. Whether such answers are satisfying is a matter of agreement. Nowadays, in the Academic world such answers would no longer be accepted [▷4] .

  An explanation assumes the presence of some accepted THEORY in which the explanation should make sense [▷5] .

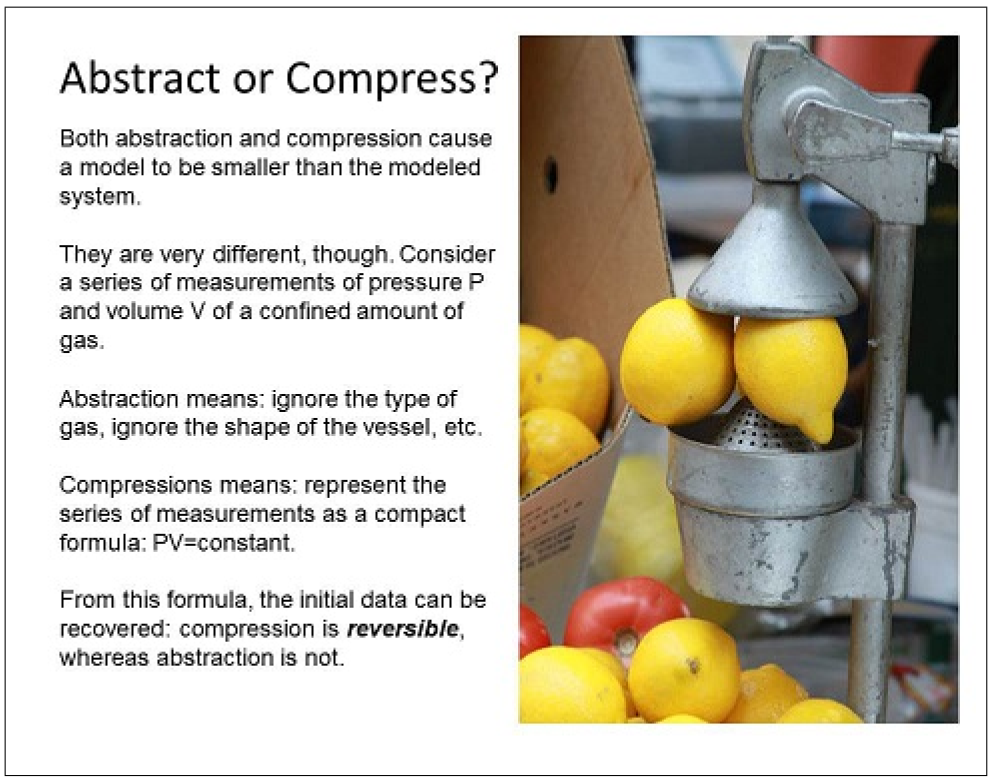
- *Ptolmaic view:* to PREDICT[3] X is answering a question of the form 'When will X happen?'

[3] The figure of a hand used for divination (predicting the future) is taken from http://commons.wikimedia.

or 'What will happen to X?' or 'When will X happen if ...' or 'What will happen to X if ...'. The four types of prediction are explained in the Scheme 'A Handful of Predictions'. Whether an explanation is acceptable is SUBJECTIVE and depending on who the explanation is intended for. A *prediction*, on the other hand, can be OBJECTIVELY verified.

- *Keppler's laws:* to COMPRESS[4] X is answering a question of the form 'How can X be written down in more compact form?'. Tycho Brahe's results comprised lengthy, numerical tables. The three concise Keppler's laws fit on a single Web-page, and contain the same information, in the sense that all of Brahe's data can be reconstructed with Keppler's formulas. In itself, the compression does not add to the understanding: Keppler does not give an answer as to *why* planets closer to the sun move faster. In that respect, compression differs from explanation. Good compression, however, can aid analysis - which in turn can lead to explanation or even prediction. Newton said that he was 'standing on the shoulders of giants': if Keppler would not have compressed Tycho Brahe's numbers into a form that was easy to apprehend, Newton might have failed to find his own results.

- *Two-body gravity:* to ABSTRACT X is answering a question of the form 'What is essential in X?'; to UNIFY X and Y is answering a question of the form 'What do X and Y have in common?'. Before Newton, it was believed that motions of falling and thrown objects on the earth and the trajectories of planets followed different rules. After Keppler's contribution, there was a useful, compact model for planetary motion, although no explanation. Newton's work added two ingredients: the compression became an explanation: now, there was an answer to the question 'why do planets, close to the sun, move faster?' ▷6 . His second contribution was to extend the scope of his model for planetary motion to the sublunary domain. The mechanism that pulls the apple from the tree is the same that pulls the moon towards the earth. This is *unification*: providing a representation that allows explanations and predictions in two domains that initially were thought to be unrelated. Unification is a special case of *abstraction*: leaving out details in the hope that the remaining essentials allow for *generic*



org/wiki/Category:Chiromancy#mediaviewer/File:Chart_of_the_Hand.png

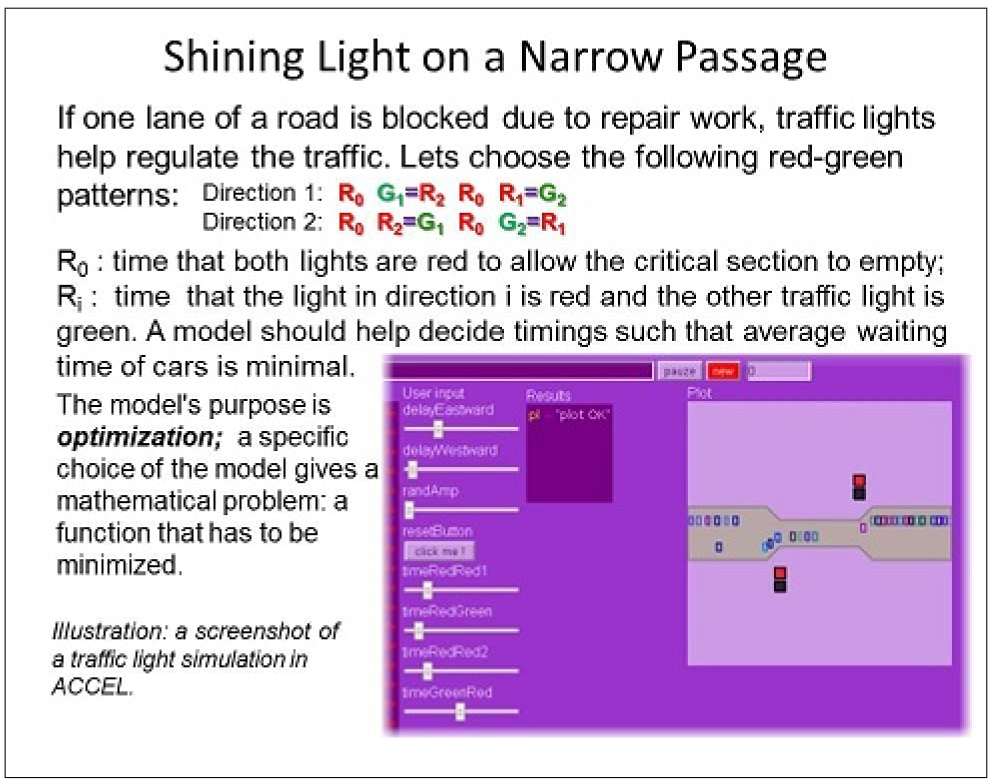
[4]The image of a lemon squeezer is taken from http://upload.wikimedia.org/wikipedia/commons/2/2e/Lemon_squeezer.jpg?uselang=nl

explanation, applying to two (or more) domains.

- *Multi-body gravity:* to ANALYSE X is answering a question of the form 'What is the relation between X and some Y?'. With today's super computers, numerical experiments are being performed to find out if we should worry about future cosmological disasters. These models are used for *predictions*; ideally, however, we would like to use them for the purpose of *analysis*. Analysis is a broader category of purpose: its aim is to DEDUCE properties of a system under study, not studying merely numerical outcomes, but rather the *laws* and *regularities* that underlie these. An analysis contributes to the *understanding* of governing mechanisms or principles; hence, on the basis of an analysis it is often possible to do predictions. Indeed, to do plausible predictions some form of analysis is typically required; not all forms of analysis, however, lead to predictions. One may analyse a single, unique phenomenon ('what was the course of events that led to the 1912 Titanic disaster'), where such analysis will not give rise to any predictions at all.

1.2.2 Purposes of Modeling in the Context of Design

The Solar System can be studied; perhaps it can be partially understood, but we cannot intervene with it. This contrasts with ARTIFACTS such as machines, tools or organizations. These are the result of human decision making: there is an intention, for instance: creating VALUE, and this intention should be fulfilled by doing adequate things. The examples in Section 1.1 fall in this category: the values to be created in the festival case are entertainment, safety for the public and profit for the sponsors and organizers. The values to be created in the wallet-example are the ability to buy food and, if possible, the possession of a particular book. The decisions are, respectively, whether or not the festival should be canceled because of weather conditions, and whether or not the book should be bought.



There are different ways to do design. Some design processes comprise the following stages; for every stage, we indicate the associated modeling purpose.

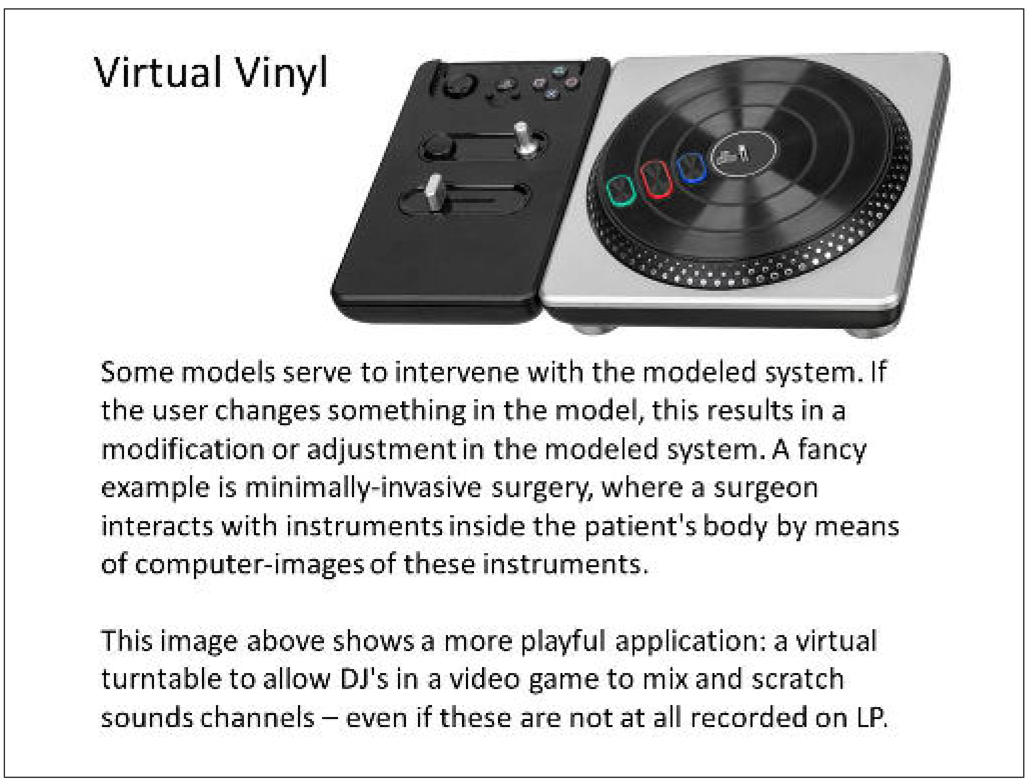
1. *Context information:* The designer needs to understand the context of the artifact-to-be-designed (we will abbreviate this with ATBD). A model is sometimes used to represent this

context. The purpose of this model is *data compression*: a large and heterogenous amount of information should be condensed into a structured and accessible format.

2. *Requirement elicitation:* this activity may result in a user requirement document. This document describes the ATBD as the user will perceive it, and as such it is a model. The purpose of this model is to COMMUNICATE ▷7 the intentions of the user to the designer. To *Communicate* X is answering a question of the form 'How do I make sure person P knows X?'. A second purpose, related to requirement elicitation, is to SPECIFY. Specifying X is answering the question 'Which requirements should X (which does not yet exist) fulfill?'.

3. *Option generation:* the elementary activity in a design process is taking a decision. A decision can have two or more outcomes. An example of two outcomes is: 'do intervention X or omit it' (a yes-no decision). With more than two outcomes, we distinguish CLOSED and OPEN decisions. *Closed* means that the list of possible outcomes is fully known beforehand. *Open* means that the designer has to invent the alternatives. A model may be used to aid in suggesting options in an open decision. This is called EXPLORATION. To explore X is answering the question 'What are the various possibilities for X?' ▷8 .

4. *Selecting the most appropriate option:* once a number of possible options is found ▷9 , the designer should select one. Often this relates to some form of optimality. The modeling purpose involved is OPTIMIZATION. To optimize X is answering the question 'What should we choose for some Y such that X is as large or as small as possible?'. A more general purpose is to DECIDE X, that is answering the question: 'What do I choose for X such that some condition is fulfilled?'.



5. *Verification:* once the ATBD is realized, it may be necessary to check that it indeed satisfies the initial list of requirements. Such models intend to VERIFY X, that is to answer the question 'Is it true that X holds?'.

6. *Steering and control:* Some artifacts, when realized, need no further models to operate. Many modern artifacts, however, employ models for their operation. Think of user interfaces for machines or software[5]. This user interface presents certain options for intervention (sliders,

[5] The image of a virtual turntable is taken from http://upload.wikimedia.org/wikipedia/commons/7/71/

buttons, ...). Conversely, the state of the artifact may be represented by dials, numbers on a display etc. They allow to STEER or *control* the modelled system, that is: affecting the status quo in the modeled system. To steer X is answering the question 'What interventions should take place in X such certain conditions hold?'. Think of the screen of an air traffic controller. This contains moving dots, each dot representing an aircraft. The operating controller may use a cursor to select or interact with one of the dots in order to instruct the respective pilot. The display, and its accompanying software form a model with the purpose to control or steer aircraft. This is an example where the steering or controlling model has a human being in the loop. Other examples exist where the steering or controlling is fully automated, e.g. a thermostat.

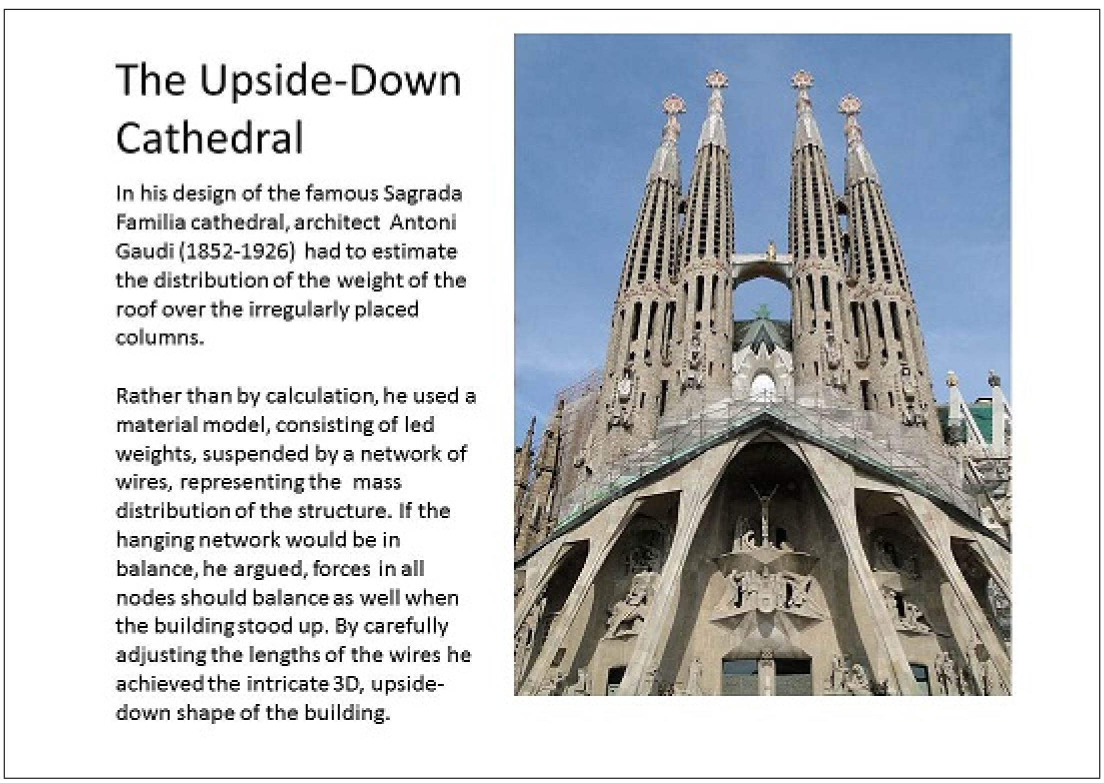
7. *Training:* Some models, such as flight simulators, driving simulators or simulators for delicate equipment serve to TRAIN prospective operators. Training to use X is answering the question: 'How could a prospective user learn to operate X without actual dealing with X?'. This includes the domain of medical or surgical training where the modeled system is (part of) a patient.

In Table 1.1 we summarize the various purposes for modeling.

1.3 Modeling Approaches

To meet their purposes, MODELs come in various kinds. We identify a number of DIMENSIONS that can be used to categorize models.

1.3.1 Material - Immaterial



To estimate[6] the forces on a vessel (say, a tanker, a freight ship or a ferry boat) due to water friction, a scaled-down copy of the vessel is sometimes built, and towed in a water tank. Measuring the water

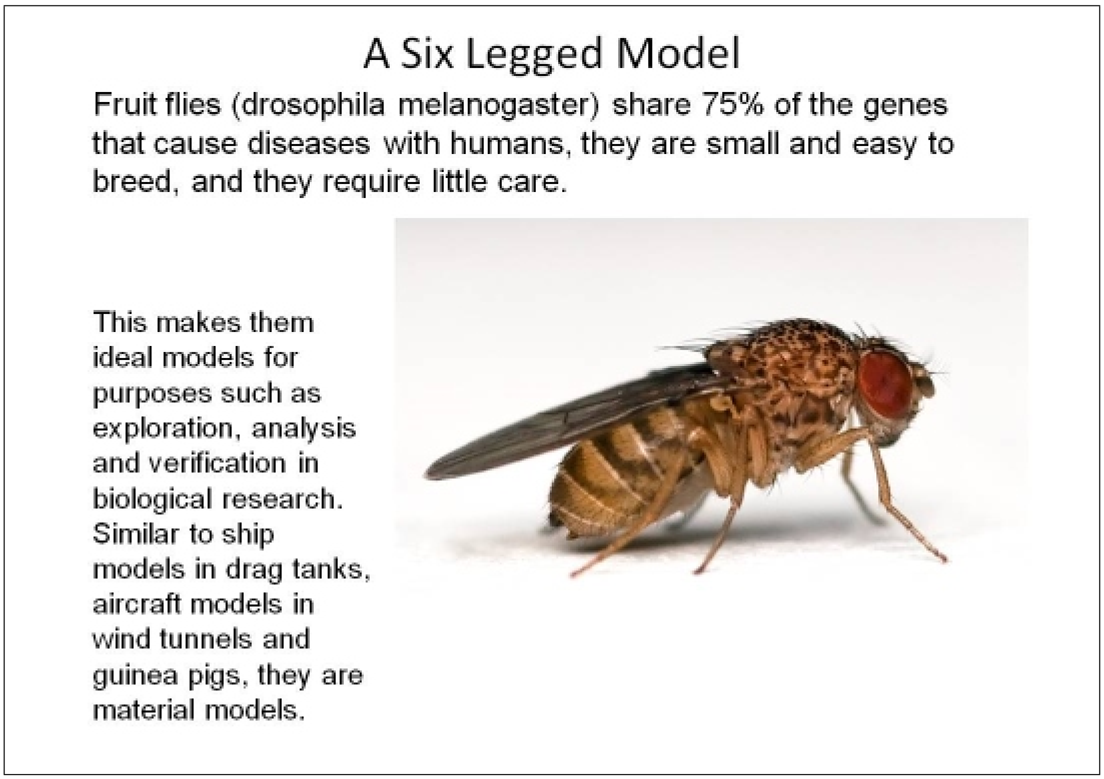
DJ-Hero-PS3-Turntable.jpg?uselang=nl

[6]The image of the Sagrada Famila cathedral was taken from http://commons.wikimedia.org/wiki/Sagrada_Familia#mediaviewer/File:Sagrada_Familia_03.jpg

Purpose	Typically found in Research (R) or Design (D)	Relevant questions
explanation	R	Who should be convinced? How non-obvious is the phenomenon to be explained?
prediction 1 ('unconditional')	R & D	What do we assume to stay the same until the prediction is to be fulfilled? How accurate should the time be foretold?
prediction 2 ('conditional')	R & D	What do we assume to stay the same until the prediction is to be fulfilled? What is the condition, and how can it vary?
compression	R	Is the data set sufficiently coherent to expect something meaningful (no outliers, no different phenomena in one data set, ...)? In what form whould the compression represent the data (formula, graph, ...)?
abstraction	R	Abstraction means: leaving out details. Are the details to leave out sufficiently irrelevant? Are the statements (predictions) about the abstraction still sufficiently concrete to be relevant for the initial phenomena?
unification	R	Why, and to what extent, should the phenomena to be unified, be similar? Are the statements (predictions) of the unified model still sufficiently concrete to be relevant for the initial phenomena?
analysis	R	What do we hope to learn from the result of the analysis?
verification	R & D	Is the route to verification independent from the route that led to the construction (e.g., verifying is a computer program is correct by just reading the code is not very helpful)? If no counterexample is found, what does this prove?
communication	R & D	To whom is the communication directed? What does the receiving party know? What does the receiving party need to know?
exploration	D	Is the collection of alternatives to be produced sufficiently broad (varied, complete, ...) ?
decision	D	Is the set of alternatives known? Is it a closed or open set?
optimization	D	How optimal does it *need* to be? Can it be assessed if the solution is or is not optimal? How much effort can be saved by going for sub-optimal?
specification	D	Which details can be left unspecified? Can the specification be realized?
steering, control	D	What sorts of perturbations should the controller be able to accommodate?
training	D	What sorts of scenario's should the trainee be exposed to?

Table 1.1: Purposes for models

drag, and scaling up the measured drag, thereby taking the spatial dimensions of the real ship into account, is a routine approach to obtain quantitative results that may be more reliable than solving approximate equations. In another application domain: to predict the effect of new medication on the human metabolism, guinea pigs may be injected with the experimental drug, their physiological reactions being interpreted in terms of risks to the human body.



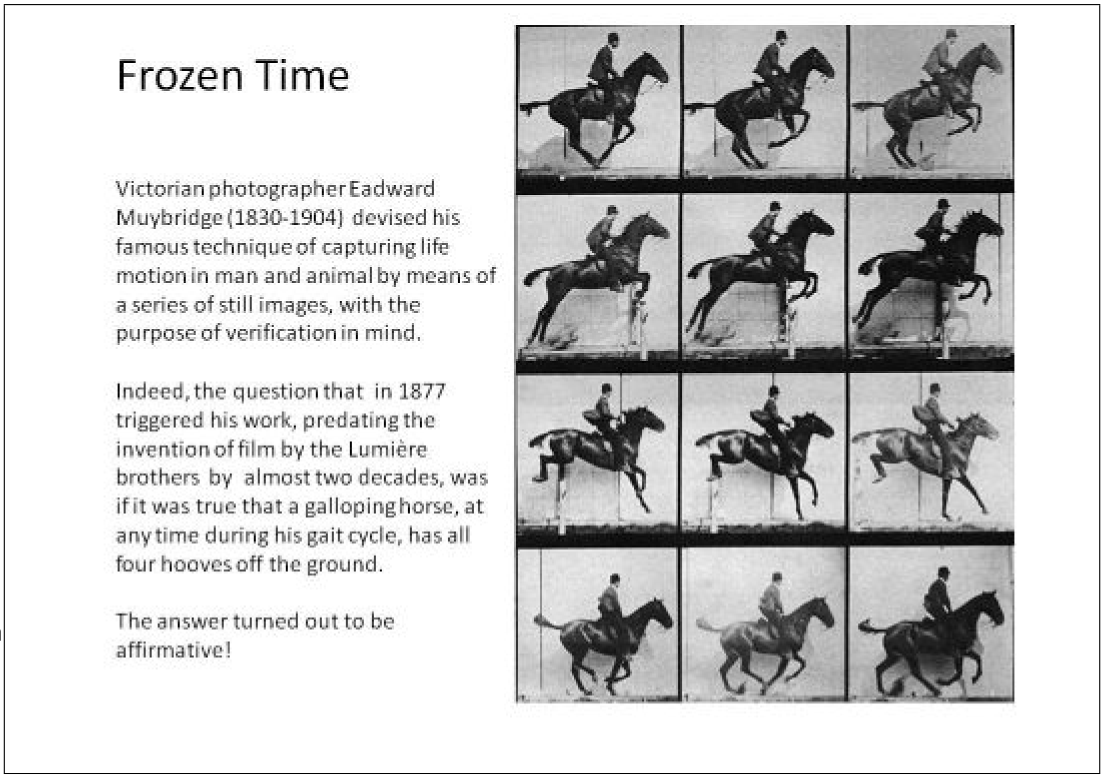
Scaled-down vessels and guinea pigs[7] can act as part of a model. They need a non-material context, however, in order to fulfill a purpose. The measured forces on the scaled-down vessel need to be converted into forces acting on a (hypothetical) full-scale vessel using dedicated mathematical transformations. The guinea pig experiment needs to be repeated a number of times in order to be statistically significant, and both doses and symptoms need to be translated to the case of human anatomy in order to mean something. Still, for the sake of brevity, models involving material objects are sometimes called 'material models'. The object occurring in material models can be both a natural object (like a guinea pig) or an artifact (like a scaled-down vessel).

MATERIAL MODELS contrast with IMMATERIAL MODELS.

Immaterial models contain no other material objects than the carriers of information (e.g., paper and ink, or computer memory). Immaterial models are, for instance, mathematical models, consisting of equations and functions; logical models, consisting of facts and rules to connect facts, or software models, consisting of computer instructions, for instance to drive a computer simulation. These three are commonly called FORMAL MODELS or formal systems ▷10 , see also Section 1.4.3 and Chapters **??** and **??**.

[7] The photograph of a fruit fly was taken from http://commons.wikimedia.org/wiki/Drosophila#mediaviewer/File:Drosophila_repleta_lateral.jpg

1.3.2 Static - Dynamic



Some problems require time to be taken into account[8], and some don't. To know how strong an electric lamp needs to be to illuminate a segment of motor way at night one can safely ignore the PROCESS of illuminating (that is, light waves traveling through space, reflecting, etc.). It is allowed to talk about STATIONARY quantities, such as light intensity, electric power, reflection coefficients, etc. On the other hand, in a model that predicts how long something takes (say, cooking a 7 kg turkey in a 250 °C oven), it is obvious that the quantities occurring in the model will depend on time. Models that involve time are called DYNAMIC; the opposite is called STATIC.

Many quantities depend on time, but time does not depend on anything else ▷11 . Also, time-dependent quantities may depend on earlier values of the same or other quantities (*memory*), but not on a *future* value of any other quantity. Chapter 3 is entirely devoted to dynamic models.

1.3.3 Continuous - Discrete

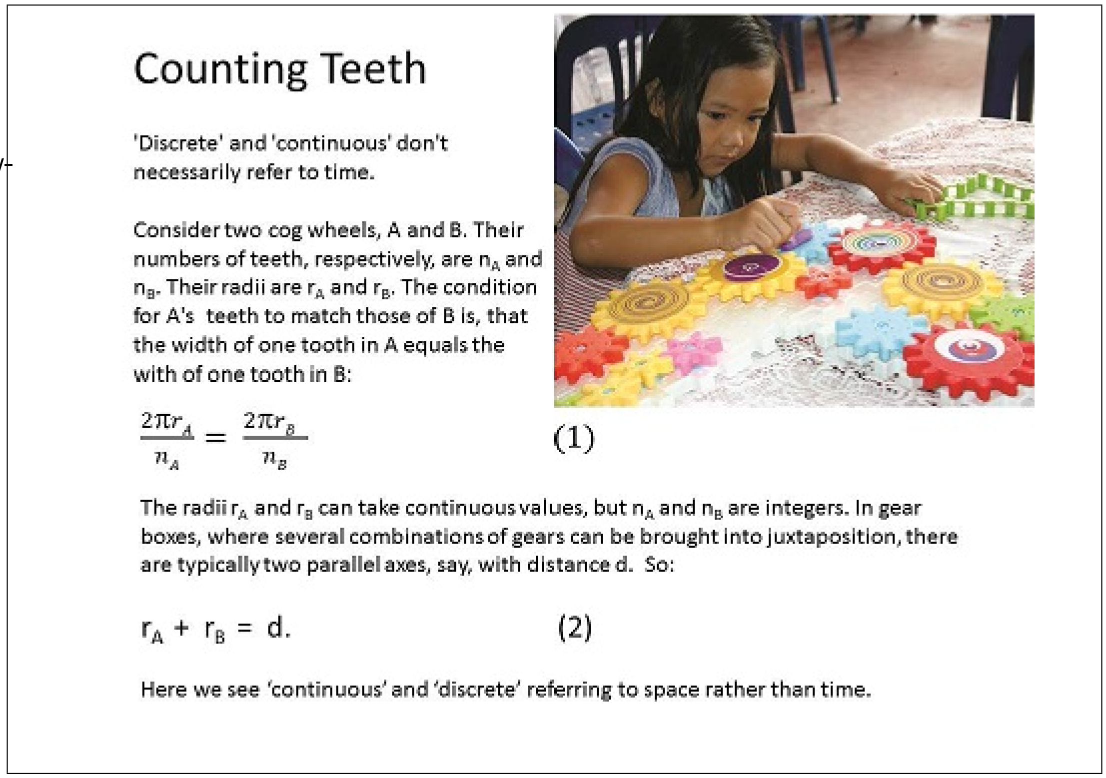
All models involve quantities. Quantities often correspond to sensorial impressions. We see a raindrop gliding down on the window pane, and the vertical position h_r of this raindrop assumes *all* values between the top, h_T, of the window and the window sill, h_{WS}. We see that h_r passes through *all possible values* in between h_T and h_{WS}. It doesn't skip any location. Even if we would follow the raindrop with a magnifying glass, we would see h_r to occupy *all* values between h_T and h_{WS}. The property corresponding to 'not even skipping the smallest hole' is called CONTINUITY. A quantity, like h_r, that can assume a range of values without even the smallest hole is called CONTINUOUS.

Not all quantities are continuous. If I am in a room, I can count the people in the room.

[8]The Muybridge image was taken from http://upload.wikimedia.org/wikipedia/commons/b/bd/Muybridge_horse_jumping.jpg?uselang=nl

COUNTING means: making a correspondence between entities in a set and numbers 1,2,3, $\cdots$, assuming that for every entity, it is clear whether or not it belongs to the set. So: there is nobody standing in the doorway, no people showing on photographs, or other dubious cases. The highest number en*count*ered is the number of people in the room. This will be an INTEGER number: 0, 1, 2, 3, $\cdots$, but never 4.7 or 5.3. Quantities that can only occur as integer numbers are called DISCRETE.

Many problems involve continuous quantities[9]. To do something meaningful with continuous quantities, however, we may have to resort to SAMPLING. Sampling approximates a continuous quantity by a discrete quantity. This discrete quantity occurs in steps that are so small that in practice no problems occur when working with the discrete quantity instead of the continuous one.



The Lumière brothers, when they invented motion picture in 1895, introduced sampling of moving scenes. A money system, using a smallest coin as unit, samples wealth.

Sampling, which will be discussed further in Section 3.3.3, is admitted if the behavior of the HYPOTHETICAL continuous quantity can be RECONSTRUCTED from a set of discrete samples. To find the height y at time t of a football from two images taken at times t_0 and t_1 in a film, showing the ball at heights y_0 and y_1, respectively, we may do LINEAR INTERPOLATION, that is calculate:

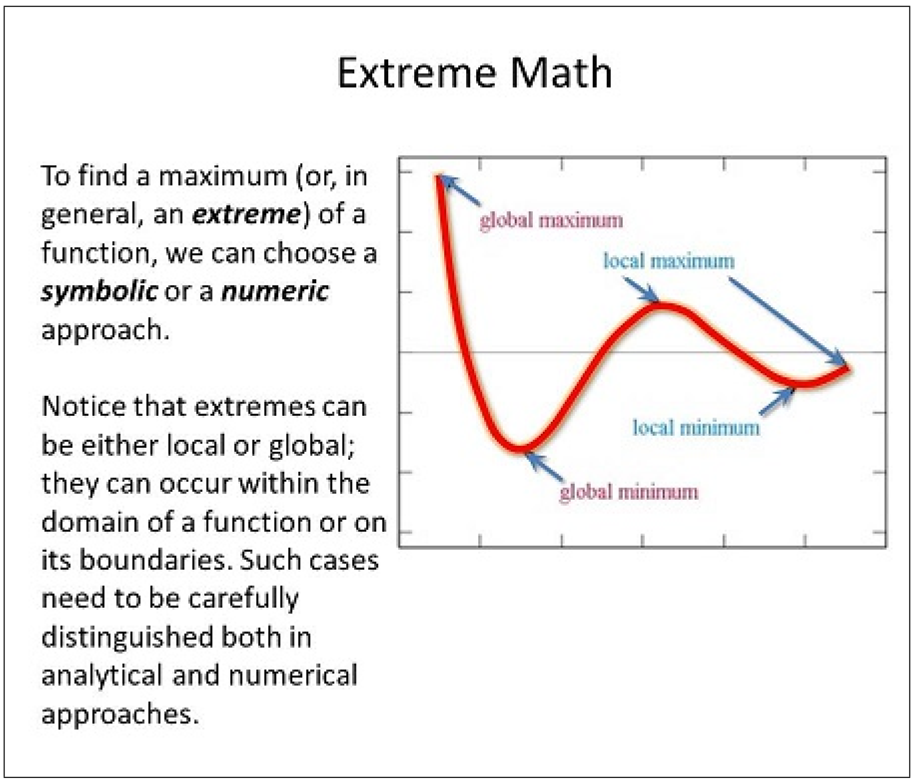
$$y = y_0 + \frac{t - t_0}{t_1 - t_0}(y_1 - y_0). \tag{1.1}$$

This is a simple example of reconstruction, where a continuous quantity (y) is reconstructed from a set of discrete samples of this quantity (y_0,y_1). This occurs in many modeling contexts ▷12

[9]The image of toy gears is taken from http://upload.wikimedia.org/wikipedia/commons/1/18/Gear_toy_and_young_girl.jpg?uselang=nl

1.3.4 Numerical - Symbolic

To illustrate the difference between numerical models and symbolic models, we look at an example. We want to know, for some function $y = f(x)$, for which value for x between x_0 and x_1, y reaches its maximum. We can reason as follows. We solve $\frac{d}{dx}f(x) = 0$ for x, and record all solutions x_i between x_0 and x_1. These correspond to so-called STATIONARY POINTS of f: points where it has a horizontal tangent. We only record x_i where f has a local maximum. Then, we evaluate $y = f(x_i)$ in all x_i-values; we also evaluate $y = f(x_0)$ and $y = f(x_1)$, and the largest of all these y-s is the answer.



This procedure uses mathematical ANALYSIS. That is: it performs operations on symbols and formulas, e.g. by differentiation, rather than numbers. It applies to any differentiable function f - but in many cases we won't be able to solve $\frac{d}{dx}f(x) = 0$. For instance, if f is a high degree polynomial, or a complicated transcendental function .

An alternative procedure is that we program a computer to evaluate $y = f(x)$ in, say, 100000 EQUIDISTANT values for x between x_0 and x_1, and store these y-values. The y-value which is the largest is a likely approximation of the absolute maximum. That is: we apply a simple ITERATIVE ALGORITHM.

Most computer programs ▷[13] operate on numbers rather than non-numerical symbols. This approach is therefore called NUMERICAL modeling.

The two strategies both frequently occur in modeling. They have a number of striking differences, though (see Table 1.2) ▷[14] .

1.3.5 Geometric - Non-geometric

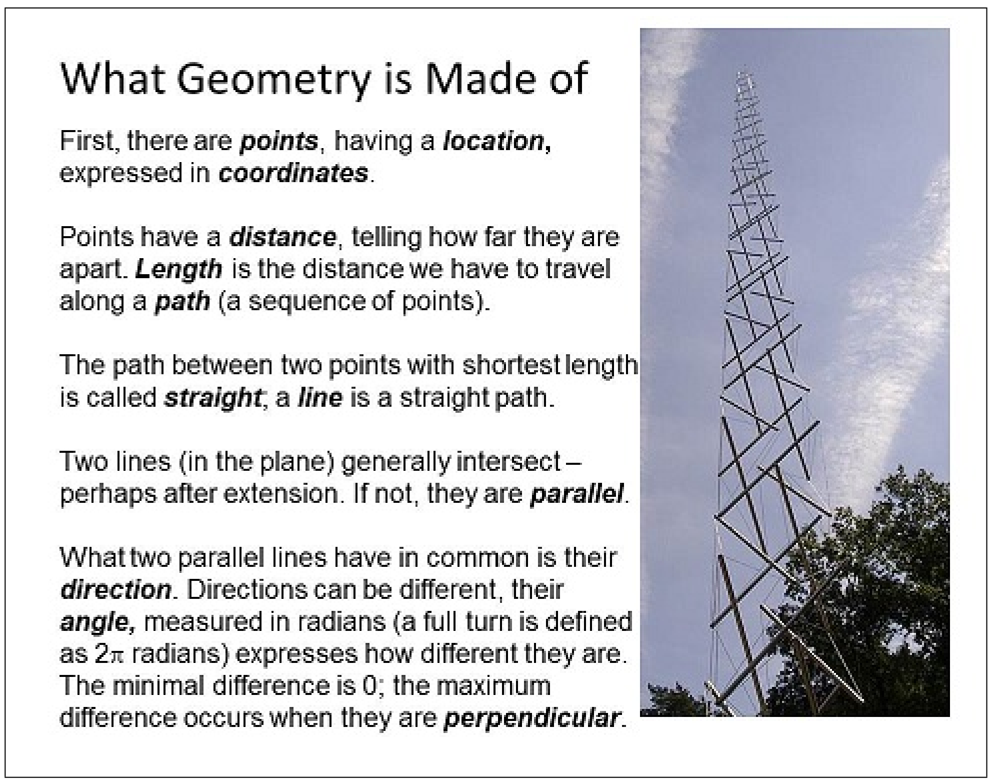
According to our sensorial experience, three independent quantities relate to space, for instance horizontal distance, vertical distance and depth distance, measured from some given reference point and using a set of reference directions. Quantities used to distinguish spatial locations are usually called COORDINATES. Many interesting problems are governed by fewer than three spatial coordinates. For a train on a track, for instance, only one spatial coordinate may be relevant,

Feature	Symbolic	Numerical
precision	irrelevant, but if the answer needs to be a number, we need numerical approximations at the end of the calculation.	a point of concern: numerical calculations typically introduce round-off errors that can make the outcome fully unreliable.
generality	limited. Many mathematical results exist, but in order to actually calculate something one is typically restricted to few simple cases. For instance: linear functions, low-degree polynomials, or trigonometric functions.	generic. Numerical operation is not restricted to closed form 'simple cases'.
ease of use	proficient use of mathematical analysis requires abstract thinking and precision, plus knowledge of textbook mathematical results.	many standard, reasonably robust methods are available, for instance in libraries (Matlab, Excel and others).

Table 1.2: Differences between symbolic and numerical modeling

as the train can not move sideways nor up and down. For a game of chess, two coordinates are enough. From the latter example, we learn that spatial coordinates can be either continuous or discrete.

Apparently, there are some fundamental concepts (DISTANCE ▷15 , STRAIGHT, LENGTH, LINE, COPLANAR, PARALLEL, PERPENDICULAR, DIRECTION and ANGLE), related to our perception of space, that have been given a more precise mathematical definition[10]. They have become MATHEMATICAL OBJECTS ▷16 . The area of mathematics that studies the relations between the sorts of objects mentioned above is called GEOMETRY.



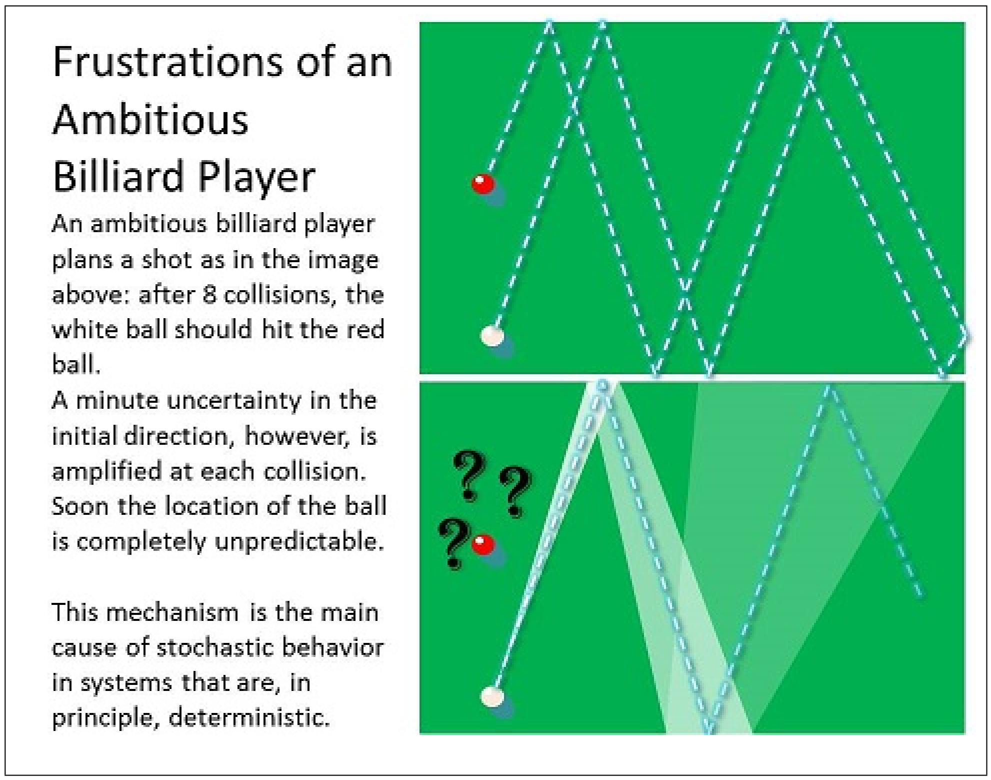
Models hinging on space-related quantities are called *geometric* models. Often, space-related quantities are properties of material objects. For instance, in the design of an artificial heart-valve, both the shape of the heart and the spatial patterns of the blood flow need to be accurately represented. Similar in the design of machine parts (3D geometry) or urban planning (2D). Models in which space-related quantities do not occur are called *non-geometric*.

[10] The image of the Needle Tower, by Kenneth Snelson, is taken from http://commons.wikimedia.org/wiki/File:KrollerMuller_ParkSculpture4.jpg?uselang=nl

1.3.6 Deterministic - Stochastic

This chapter started with two examples of models: a weather model that predicts a thunder storm, and the wallet model that predicts whether or not I can afford to buy a meal tonight. The wallet model makes a certain statement ('if you buy a 25 Euro book, it is certain that you will have enough money left to spend 5 Euro's for a meal'), whereas the weather model does a probabilistic prediction ('there is a 95% chance that ...').

To understand this difference between so-called DETERMINISTIC and STOCHASTIC models, we study an example from physics.



Let us consider a gas, consisting of atoms in a closed container. We view these atoms as miniature billiard balls. Just as billiard balls, atoms can collide with each other, and they can collide with the walls of the container. First, suppose that there are very few atoms. Just as a proficient billiard player can determine what routes the balls will follow by careful observation and control of the initial configuration, the trajectories of these few atoms could, in principle, be fully known from their initial configuration as well. The system is behaving in a deterministic manner: the outcome leaves no room for uncertainty if the initial configuration is known with 100% accuracy ▹17 . In practice, however, small uncertainties are unavoidable. Still, a trained billiard player can perform a shot resulting in two or sometimes even more planned collisions with each of the balls (caramboles) with a large chance of success. This is increasingly difficult if the number of balls increases, or if the time period during which the balls roll increases. Unavoidably, uncertainty creeps in, and soon the pattern of collisions becomes completely unpredictable or RANDOM. Compare this to rolling a die: there, also, a large amount of collisions, each time causing a change in the position and orientation of the die, makes the outcome unpredictable.

In the case of the gas, every collision of an atom with the wall of the container gives a small transfer of momentum to the container, and, similar to the randomness of collisions with sufficiently many billiard balls, these events of momentum transfers appear in an unpredictable sequence.

The predictability decreases if we increase the number of atoms. As we say, we make the ENSEMBLE bigger. The time sequence of impacts on the walls is increasingly random, but at the same time the *percentual* fluctuations in the number of impacts per second will vary less and less with increasing ensemble size [▷18] . Similar, the percentual fluctuations in transferred momentum per second will get smaller.

By the time the amount of atoms approaches a couple of million [▷19] - which is still an extremely tiny puff of gas - the fluctuations in the transferred momentum are so small that we perceive the resultant effect of the momentum transfers as a force that is *constant* in time: it is the *pressure, excerted by the gas* on the walls of the container. So, despite the inherently stochastic nature of the collection of swarming atoms, there is a property of the system *as a whole* that is not at all stochastic. This is an example of a BULK QUANTITY of the ensemble. We have LUMPED the many properties of the individual atoms (their locations and speeds) together, to find a new, so-called EMERGENT [▷20] quantity that is not at all stochastic: the *pressure*. Indeed, the pressure P of n moles of a gas in a volume V at temperature T behaves in a deterministic way, as described for instance by the gas law $PV = nRT$ (R being the so-called gas constant).

In this example we see how AVERAGING-OUT of the individual properties of a sufficient number of elementary entities in a stochastic ensemble may lead to determinstic behavior of bulk-properties[11]. Most deterministic laws of nature present themselves in this way: they are deterministic, because they relate bulk-properties, applying to ensembles of sufficient size. Also outside physics, emergent behavior and bulk-properties occur: think of demographic phenomena, economy, traffic, communication and many other fields.

Why the Bank Always Wins

Monte Carlo methods are mathematical methods, to estimate the value of quantities that involve chance. They get their name from the city of Monte Carlo, the *Capital of Casino*.

Every turn of the roulette wheel has un uncertain outcome, perhaps bringing fortune to the players. The odds that the bank will win in a pair-impair or black-red gamble are only 17/(16+16) = 0.531.

Despite this small advantage, and despite chance: in the long run, the bank wins with certainty, spelling doom, on average, for any gambler.

There are modeling strategies based on this idea of ensembles. In such a model, sometimes called MONTE CARLO MODEL, the purpose is to obtain emergent behavior out of a sufficiently large repetition of a simulated experiment, where individual variations are assumed to be random and UNCORRELATED.

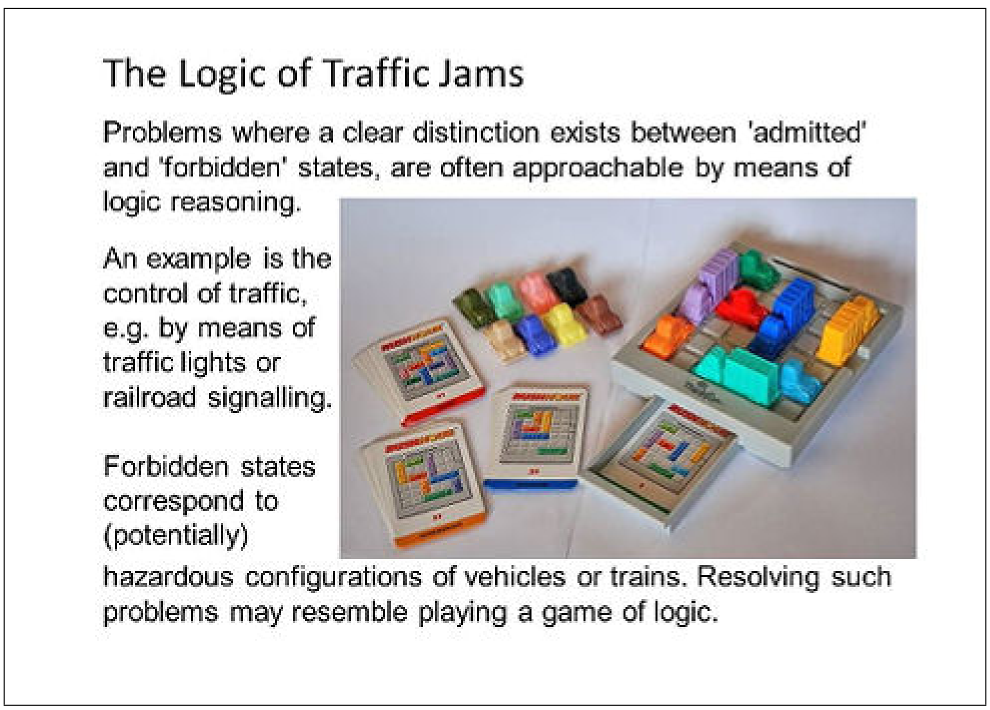
For example, we may want to find out if eating apples is healthy. We take two equal sized groups of people, one group of apple-eaters and the other of people who don't eat apples. Next we ask everybody for the numbers of times (s)he visited a physician over the last three years. Obviously, the numbers vary from individual to individual, but if apples are healthy we expect that, apart from the random variations,

[11] The roulette table image comes from http://commons.wikimedia.org/wiki/Roulette#mediaviewer/File:13-02-27-spielbank-wiesbaden-by-RalfR-064.jpg

there is a significant difference between the averages in the two groups.

Lumping omits details in individual entities in models of stochastic systems. Also in deterministic models, details may be deliberately omitted. For example, consider the traffic lights problem discussed in Section 1.2.2 (figure 'Shining Light on a Narrow Passage'; go to this link for an interactive demonstration). Here the flow of traffic has to be modeled to solve the problem. Typically, random components are present in the flow of traffic and a stochastic model lies at hand. A deterministic model, however, might be useful to get some knowledge concerning the problem and might be good enough for the given purpose. One might assume, for instance, that the cars arrive in a deterministic pattern with a constant inter arrival time. So the variability in the traffic flow is not accounted for in this model and detailed knowledge is omitted by deliberately ignoring the randomness in the modeled system.

1.3.7 Calculating - Reasoning



Many models involve *calculations*, either with numbers or with symbols. There is a different class of FORMAL models, however, where answers do not come from operating upon numbers[12].

For instance: consider a software system to control signals and switches (Dutch: 'wissel') of a railroad junction. As we will discuss in more detail in 3.2.1, the combination of all states of the signals and the states of the switches can easily amount to many billions of possible combinations. Many of these are illegal: for instance, if the signals on both incoming branches of a switch show green, we can foresee a collision if two trains approach this switch at the same time. It is impossible, though, to verify all states by hand to see which are admitted. So it is impossible to verify if the software leads to safe train traffic by just inspecting the computer program.

Instead we make a model of the junction-control software in terms of logical expressions. By combining these expressions, using another computer program, we may be able to prove that indeed the software system is safe, or we may be able to spot flawed configurations. Logical expressions, in some sense, resemble arithmetic expressions. There are also quantities, and the values of some quantities depend on others. But instead of addition, multiplication, etc., the

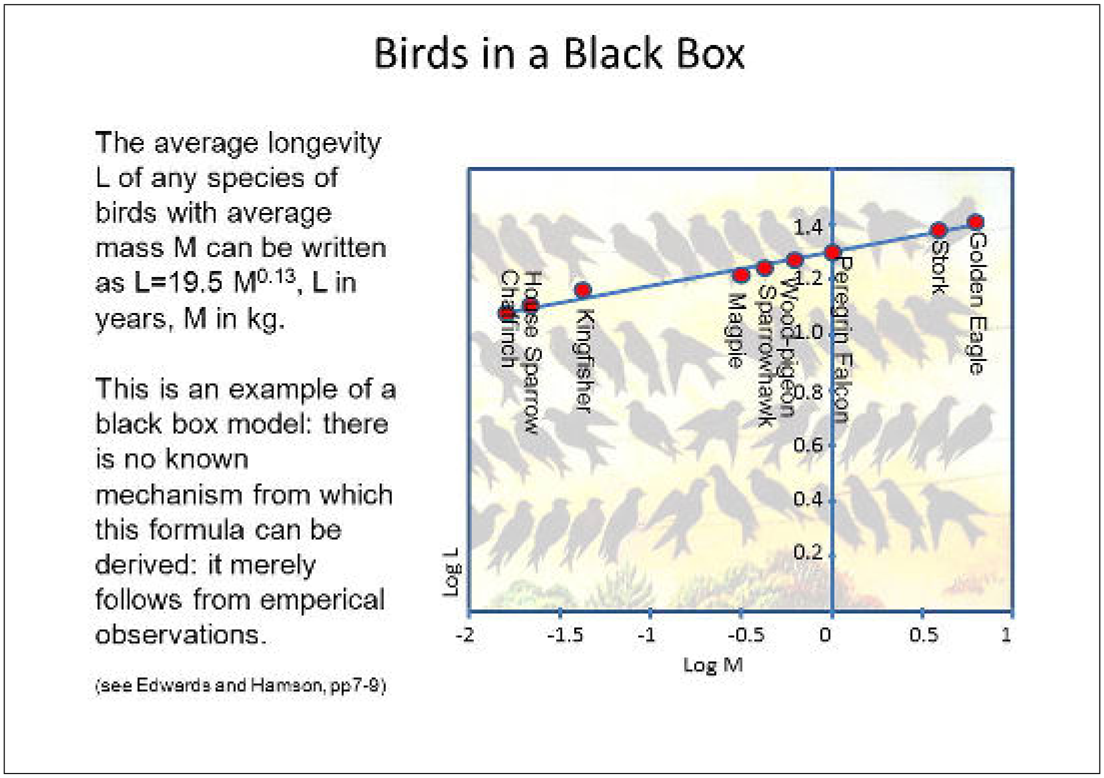
[12] The image of the rush hour game is taken from http://commons.wikimedia.org/wiki/Category:Logic_puzzles#mediaviewer/File:Rush_Hour_sliding_block_puzzle.jpg

logic expressions consist of such operations as ' AND', ' OR', ' IMPLIES', etc.; the values of the quantities are 'TRUE' or 'FALSE'. Using a logical model we can REASON rather than CALCULATE.

Other examples include the retrieval of information from DATABASES, or the application of logical rules in an EXPERT SYSTEM or KNOWLEDGE BASE.

1.3.8 Black Box - Glass Box

For a number of species of birds, we assess their longevity and their average mass[13]. We have no A PRIORI idea if these two are related. Plotting the data in a graph may help our intuition. Such a plot is a model with the purpose of *compression*: it may suggest us how to proceed. If mass and longevity would be unrelated, the data points would be scattered all over the place. The plot seems to suggest, however, a smooth dependency. Plotting the data on a log-log graph, we find that the behavior is reasonably well described as $\log_{10}(L) = 0.13\log_{10}(M) + 1.29$, for L=longevity in years and M=mass in kg. We have performed a kind of *compression* here: we start with a data set with several dozens of numbers, and we reduce this information to one formula with two FITTED quantities.

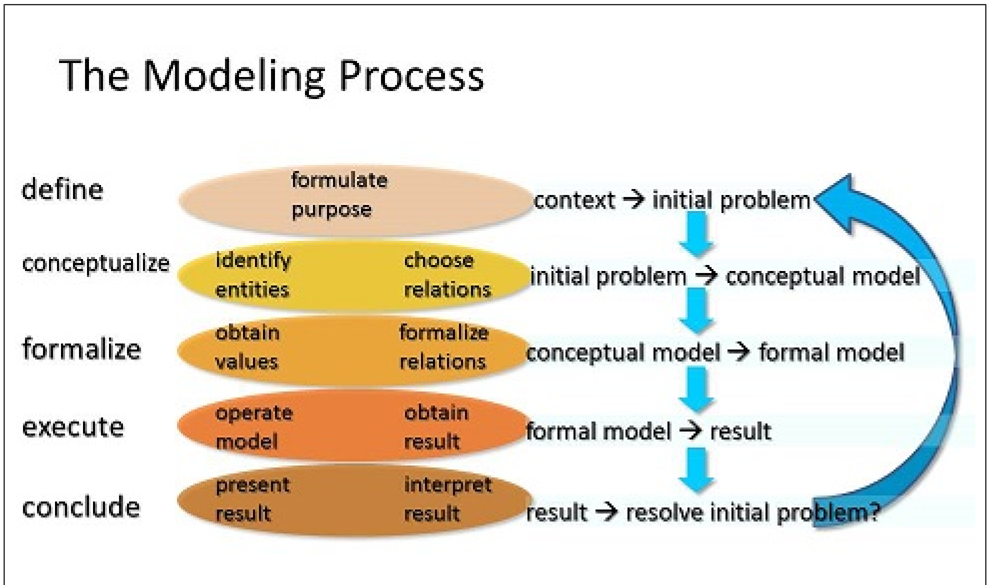


This formula says nothing about any species of birds *not* occurring in the initial data. Since we have no explanation of why longevity should be related to mass, we cannot make any claims about new species of birds. But we might want to check if the data for other kinds of birds happen to be close to the straight line in the log-log plot ▷21 . Then the compressing power of our formula improves: we can compress more data with the same accuracy with a single formula. Next, even though we have no explanatory model, we may be tempted to start doing predictions. Compare this to the Keppler approach to planet motions, or the relationship between longevity and mass in birds from the Figure 'Birds in a Black Box'.

[13]The oil painting of birds on wires is taken from http://commons.wikimedia.org/wiki/File:Birds_on_a_Wire.jpg?uselang=nl

This type of modeling is called BLACK BOX modeling. The idea of a black box is that the inside is invisible. We cannot see 'inside' the biological mechanisms that cause a certain species of bird to attain a certain mass, or that causes them to reach a certain age. Still, the fact that the data points are not random, but allow compression, suggests that there *is* a connection between these two mechanisms. Perhaps both are caused by some, hidden, third mechanism. The stronger the compression, the stronger the suggestion that there is a deeper mechanism ▹22 hidden in the system. A black box model, however, does not attempt to reveal these mechanisms.

On the other hand, a transparent box or GLASS BOX offers an unrestricted view on the internal workings inside the box. Glass box modeling starts from POSTULATING the way the we think things behave.



For instance, an engineer investigating the behavior of tall buildings during an earthquake may postulate that the buildings behave as an upside-down pendulum. Starting from this postulate, (s)he sets up a model for such a pendulum. So: the internal mechanisms of a glass box may be completely 'wrong' (a tall building *is* no pendulum), but they are used none the less to obtain predictions or other statements about the MODELED SYSTEM.

The term WHITE BOX is sometimes used instead of glass box.

A combination of a black box model and a white (glass) box model for the same system is sometimes called a GREY BOX model.

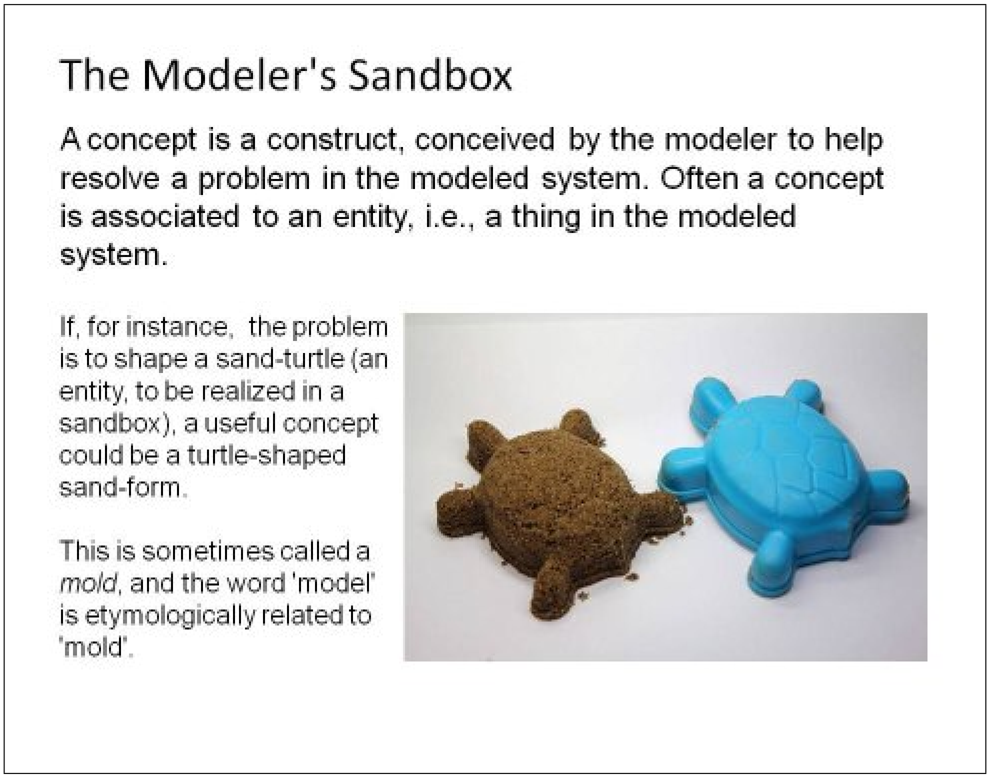
1.4 Stages in The Modeling Process

The MODELING process assumes a number of different stages, as depicted in the diagram 'The Modeling Process' ▹23 . The stages in the modeling process are called **definition**, **conceptualization**, **formalization**, **execution**, and **conclusion**. Further, after the completion of a modeling process, the modeller may do an **evaluation**. Below the stages are elaborated.

1.4.1 Definition Stage (1)

Any modeling process starts with the problem DEFINITION, in which the initial problem is stated and the purpose is formulated. The overview of purposes in Table 1.1 provides some help here, although models often should serve several purposes at once. It also may happen that, during the modeling process, the purpose changes.

1.4.2 Conceptualization Stage (2)



During the CONCEPTUALIZATION stage, the CONCEPTUAL MODEL is constructed. The conceptual model, to be elaborated in Chapter 2, is a collection of entities[14], their properties, and relations between them, but these are not yet in mathematical form. So there are no equations yet, and no mathematical derivations: these constitute the formal model, to be elaborated in Chapters **??** and **??**.

- *Conceptualization (2.1): Identify Entities and Properties*

The things that occur in the problem domain could just be called 'things', but we prefer the term ENTITIES. To some of the entities occurring in the model domain, we associate concepts in the conceptual model. The concepts in the conceptual model stand for those entities of the modeled system (i.e., the problem domain) that we want to take into consideration. The set of entities to be considered must be large enough to suit the purpose. For instance, it should contain elements allowing us to assess if the problem is solved. Also, the set of entities should not contain too many elements as this makes the model unnecessarily complex.

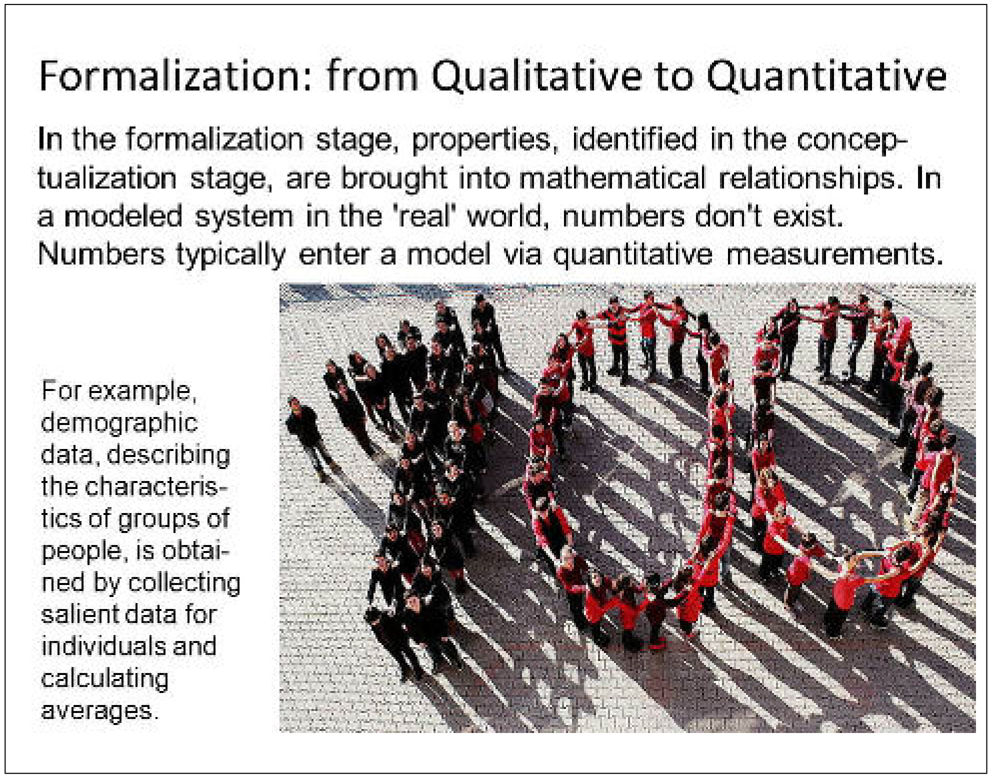
- *Conceptualization (2.2): Identify Relations between Entities and Properties*

If entities in a model are isolated, the model won't *do* anything. We need relations between entities as well. Relations are often depicted in a schematic drawing. In such a so-called GRAPH the entities are boxes or circles, the NODES. The relations between the entities are ARCS or ARROWS. The *meaning* of such arcs or arrows should be made explicit, and used consistently. We call this LABELING. This way to denote a set of entities is called and ENTITY-RELATION GRAPH.

[14]The image of a turtle mold was taken from http://upload.wikimedia.org/wikipedia/commons/7/70/Sandfoermchen-3.jpg?uselang=nl

1.4.3 Formalization Stage (3)

To 'do' mathematics (calculations) or logic (reasoning), we need quantities to compute or argue with. Many of them will represent properties of the entities from the conceptual model.



The FORMALIZATION stage transforms the conceptual model, consisting of concepts, properties and qualitative relations, to a FORMAL MODEL consisting of quantities and quantitative relations connecting them.

For quantities to occur in computations, they have to have values, and they have to take part in formal relations. An example of a formal relation, also called FORMULA is '1+1=2'. The symbols '1', '2', '+' and '=' have formally defined meanings. ' FORMAL' means: that which is defined in a logically consistent system, and does not require human interpretation in order to be operated. Arithmetic is the best known example of a formal system, but also the rules to play games such as tic-tac-toe, go or chess are formal systems.

In the formalization stage, we transform the conceptual model to one that is expressed in a FORMAL form, that is: one that no longer relies on human interpretation. There is a number of different ways values can be BOUND to quantities. One very common one, is that values come in as measured data.

Measured data[15] either can be RAW ▷24 data, or it can be processed data. Processing data often amounts to grouping a collection of numbers (=measured values) together, and summarizing the information in these measured values into a few numbers. The latter is called AGGREGATED data. Averaging, such as in Section 1.3.8, is an example of aggregation.

- *Formalization (3.1): Obtain Values for Quantities*

This stage is about the collection of data that serves as input for the model: either raw data or processed data, such as aggregated data.

- *Formalization (3.2): Formalize Relations between Quantities*

If the model is a glass box model, it contains knowledge about the mechanisms inside the modeled

[15] The image of a celebrating crowd forming the number '100' is taken from http://commons.wikimedia.org/wiki/Category:Human_formations#mediaviewer/File:100_years.jpg

system. For a black box model, these relations come in the form of HYPOTHESES, for instance that there is a linear relation between log(longevity) and log(mass) in birds, as in the example from Section 1.3.8. A hypothesis is a postulated proposition or relation, that is assumed to be provisionally true, but that will be subject to testing.

> **Recipes to Reckon**
>
> Formal operations are those manipulations to symbols (including numbers) that can take place without interpretation.
>
> For instance, the equality
>
> $$(a+b)^2 = a^2 + b^2 + 2ab$$
>
> holds irrespective of the values or the meanings of a and b.

Parallel to collecting, discovering or introducing relations, the modeler should collect, discover or introduce ASSUMPTIONS. Assumptions should be documented; they can assist later to assess or limit the plausibility of the model's result, and they can inspire to do MODEL REFINEMENT. Model refinement means: one or more iterations of the modeling process in order to obtain a model that better fulfills the purpose.

1.4.4 Execution Stage (4)

Once we have a formal model[16] that we feel confident with, we can start *using* the model for its purpose. This is called the EXECUTION of the model. In many cases, this involves some form of calculating or reasoning, e.g. to solve equations, to search for an optimal solution, or to perform some algorithm.

- *Execution (4.1): Do Operations with the Model*

The OPERATIONS with a model should comply with its intentions. These operations should lead to fulfillment of the purpose. For instance, if the purpose is 'compression', the model should produce a compact representation of data; if the purpose is 'verification', the model should produce a result 'TRUE' or 'FALSE', etc..

For some of the purposes from Table 1.1, such as 'exploration' and 'communication', there may be no need for formal operations. Think again of our black box model regarding birds' masses and longevity. If the sole purpose of the model were to communicate empirical findings, the data from a table plotted on a log-log scale could be an adequate result.

But for formal models, in general, there is at least some formal operation in the modeling process.

In some formal models this amounts to mathematical handwork. Say, deriving expressions, applying

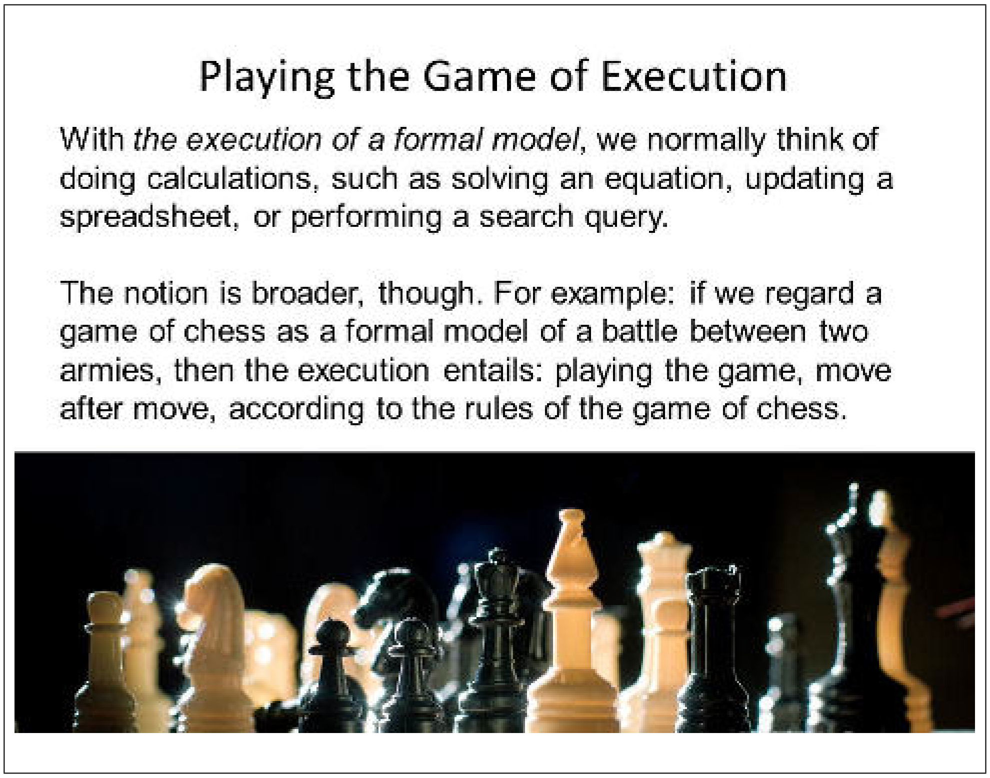
[16] The woodcut with medieval calculus-masters was taken from http://upload.wikimedia.org/wikipedia/commons/3/38/Gregor_Reisch%2C_Margarita_Philosophica%2C_1508_%281230x1615%29.png?uselang=nl

transformations, doing CALCULUS, etc.. Most formal models, however, involve a computer; in that case, *operating the model* amounts to running a computer program. The relations between quantities, established in stage (3), then lead to the statements instructing the computer to perform calculations or INFERENCES for obtaining the desired result.

- *Execution (4.2): Obtain a Result*

Most purposes from Table 1.1 cause a model to produce a result in the form of some mathematical object[17]: the value of a quantity, a set of numbers, a graph, etc.

Apart from obtaining the sought mathematical object, the modeler should always strive for obtaining insight as well. After having completed the operations with the model, looking at the obtained result, the modeler's first question should be: 'so what?'. This is elaborated further in the following stages.



1.4.5 Conclusion Stage (5)

The CONCLUSION stage comes after the execution stage. The *results* obtained from the execution are mathematical objects: numbers, graphs, and perhaps more advanced things. The purpose of the model, however, was not stated in mathematical terms. Therefore, there is always the need for a translation back to the problem domain. This translation involves PRESENTATION and INTERPRETATION of the result.

- *Conclusion (5.1): Present the Result*

The purpose of the model relates to a problem, and therefore some PROBLEM OWNER, STAKE HOLDERS, and a PROBLEM CONTEXT. For none of these, in general, the model outcome will be appealing, useful or even comprehensible.

Take for instance a model for predicting the weather. The problem owner is, say, a meteorological institute that sells advise regarding weather conditions; stake holders are people who want to know about the weather. The problem context, among other things contains people's tendency to complain about bad weather, but also to complain about good weather if it was predicted wrongly,

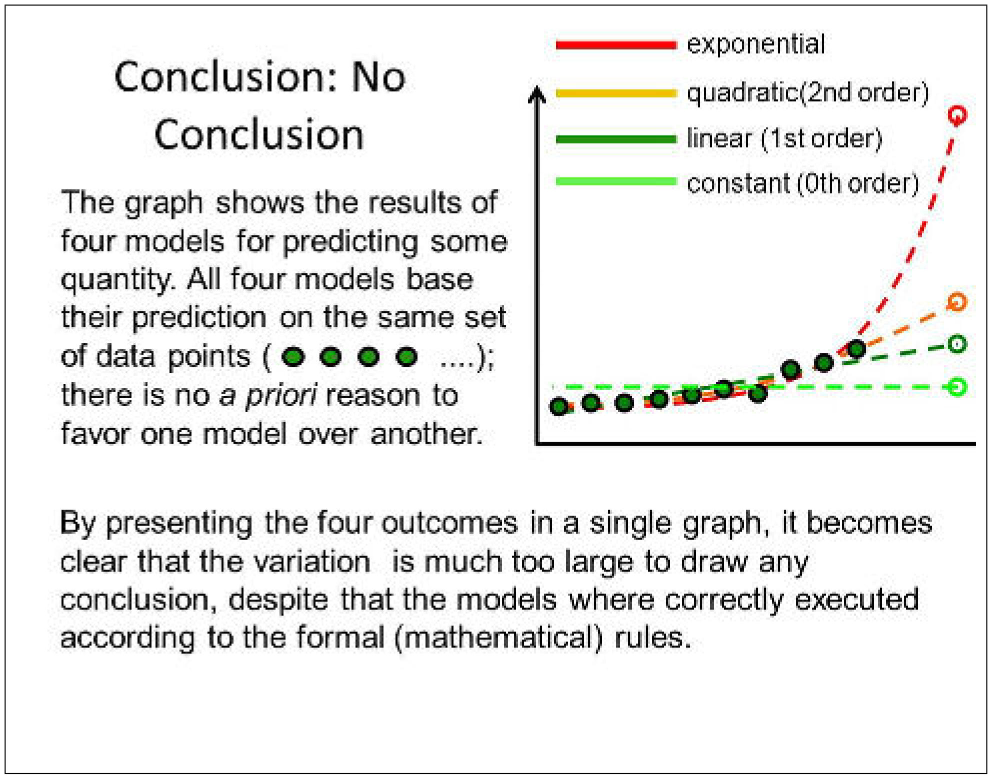
[17] The photograph of chessmen is taken from http://upload.wikimedia.org/wikipedia/commons/0/08/Chessmen_in_backlight.jpg?uselang=nl

and to have a short memory with respect to the quality of earlier predictions.

The model's outcome, obviously, is a table with thousands of numbers, say, representing predicted temperatures as a function of time and place. This table is next to useless for clients such as festival organizers, and it does not fit with the problem context. Therefore, over the last decades, there has been a considerable development in the presentation of weather maps - nowadays including computer animation and even simulated 3D effects.

- *Conclusion (5.2): Interpret the Result*

A well-thought of presentation[18] is crucial for the impact, convincingness and overall success of a modeling effort. But it may not be sufficient to fulfill the initial purpose.



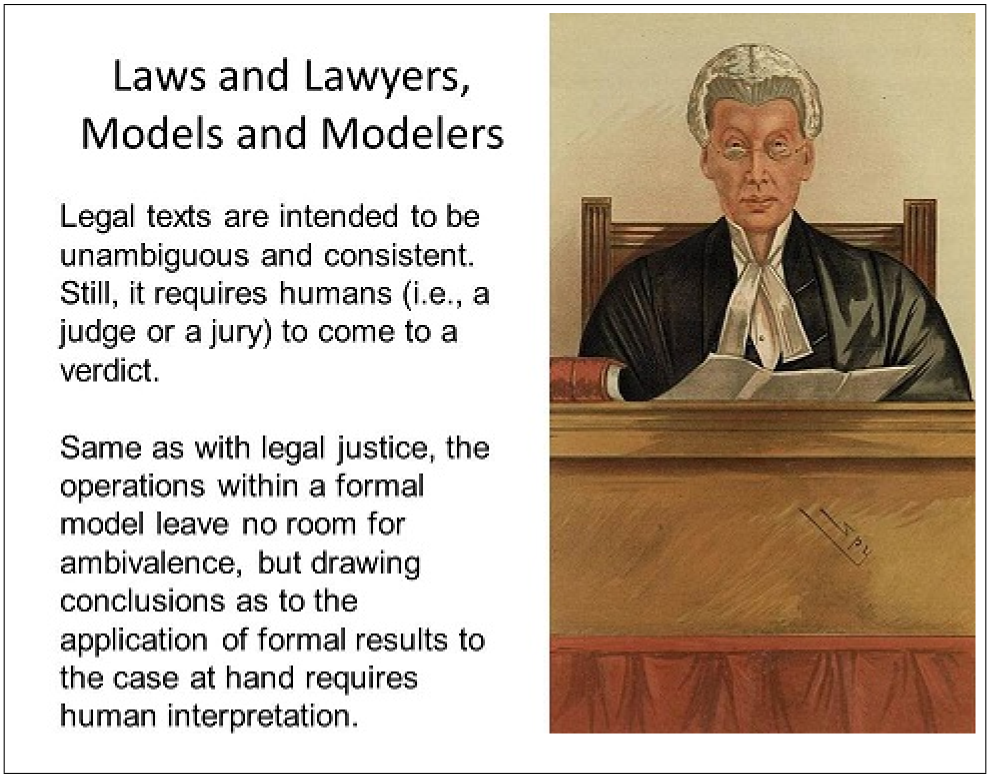
Consider the following example. A numerical model is used to calculate the concentration of some medicine in the blood in dependence of several metabolical conditions; from this concentration, the model does predictions about effectiveness and side effects of the medication. The numerical outcomes are presented using professional graphics. For patients, and even for physicians, however, this presentation has little value. Indeed, the initial question is: when should a patient take his pills? Therefore, the presentation should be interpreted, leading to answers such as: 'it is best to take two pills just after breakfast, and a third one later in the day if the fever returns', or something similar. The step from graphs and tables to this form of recommendation is called *interpretation*. It involves non-trivial skills, and it may require consulting domain specialists who have a profound understanding of the problem context.

1.4.6 Learn from what you have done: Evaluation for the Modeler

If a valid model leads to plausible results, and the interpretation of these results show that the right problem has been addressed, there is much to be proud of. There is, however, always room for improvement. Even if the current problem instance does not allow or require this improvement: for the modeler proper, pondering on this 'room for improvement' may be advisable. There will

[18] The image of a graph was based on http://commons.wikimedia.org/wiki/Category:Extrapolation#mediaviewer/File:Fstaals_extrapolatie_8.jpg

be next problems requiring next models, and a secondary purpose, hidden in every problem, is the modeler's desire to keep learning.



Therefore a last stage, devoted to REFLECTION should conclude every modeling exercise[19]. Chapter **??** is devoted to systematic techniques for reflecting on the entire modeling process.

1.5 Reflections: Iterating the Modeling Process

1.5.1 Reflection after the Definition Stage (1')

The modeling process is an ITERATIVE process. Iterations may occur at any stage. A change at the level of the identification of the purpose, however, can have drastic consequences since every subsequent step may be affected.

After the problem owner communicates his version to the modeler, the modeler will reformulate the problem in terms of purpose and context. Before going further, it is essential that the modeler goes back to the problem owner to verify if this new, redefined, version of the problem is still in accordance with the problem owner's intentions.

1.5.2 Reflection after the Conceptualization Stage (2')

Identifying the right set of entities is almost impossible to do right the first time. Revisions to the conceptual model are common in a modeling process.

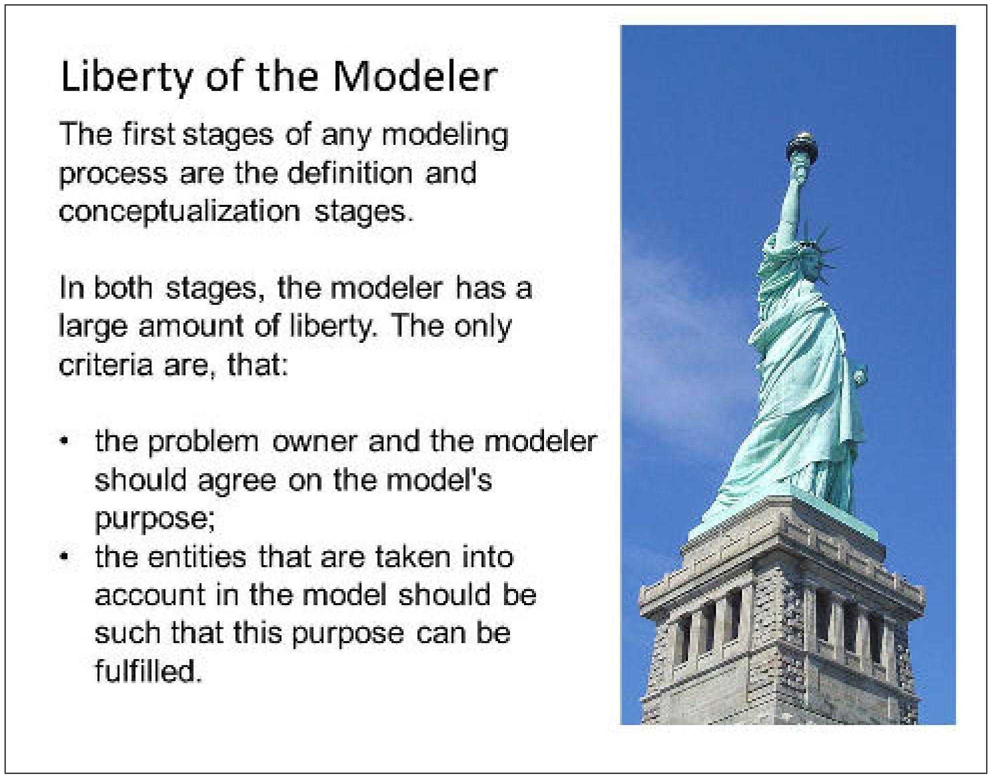
At the end of the conceptualization stage, the model cannot yet *do* anything. It cannot yet compute anything, as there are no formulas, no computed numbers, and no computer script. We have gained a fair amount of insight, though. We should check if the conceptual model reflects our intuition about the things that matter. Did we forget anything? Is there not too much detail to start with? Also, we should see if the relations that we identified are adequate for the purpose

[19] The image of a judge was taken from http://upload.wikimedia.org/wikipedia/commons/5/5b/Chitty_JW_Vanity_Fair_1885-03-28.jpg?uselang=nl

of the model - and if not, we should correct this.

1.5.3 Reflection after the Formalization Stage (3')

During formalization[20], there will come a point at which a first formal version of the model is ready. This is a good moment for a reflection. The modeler should seek arguments to support the conclusion that the formulas are good enough for the model's purpose.



This is also called the *validation* and *verification* of the model. Validation means: checking that the model produces output that is valuable for the model's purpose, for instance that it gives sufficiently small uncertainties. This typically involves running the model on input data sets for which the outputs are empirically known, and see if the model reproduces these known outputs, prior to executing the model for the actual purpose at hand. Such known outputs are called GROUND TRUTH: if the model produces output that *differs* from ground truth data, we are certain that the model is *wrong*.

Verification amounts to checking the logical and mathematical consistency of the formulas. Consistency is a necessary, but not a sufficient condition for an adequate formalization.

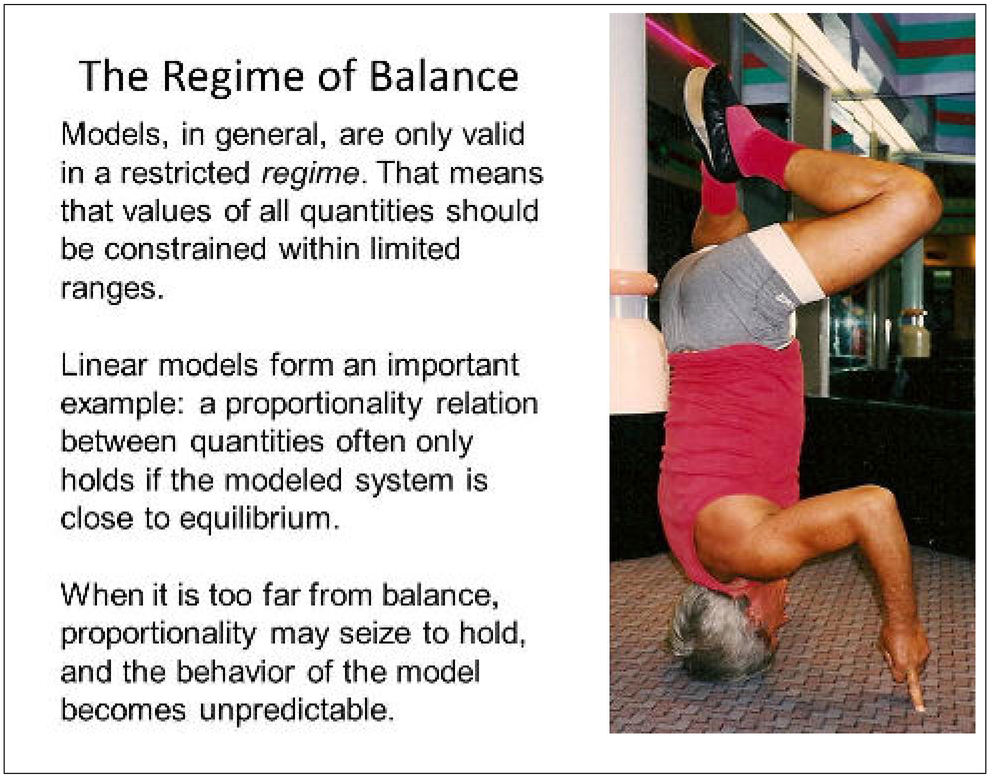
Similarly, successful validation against ground truth data is necessary but not sufficient. It is always possible that the tests with ground data miss a peculiar case. Indeed, any form of NON-EXHAUSTIVE testing resembles sampling few oranges from a full batch: even if the sampled ones are sweet, there is no guarantee that there are no sour oranges in the batch ▷[25] . In Chapter **??** we learn some techniques for validation and verification.

1.5.4 Reflection after the Execution Stage (4')

The reflection on the outcome of the execution of a model should verify if these numerical outcomes fall in the REGIMES that were assumed in the various parts of the calculations.

[20] The image of the statue of liberty is taken from http://commons.wikimedia.org/wiki/Statue_of_liberty#mediaviewer/File:Majestic_Liberty.jpg

A regime is a range of values for the quantities in a model such that the model behaves similarly, or a range of values for the quantities in a model such that the same set of assumptions hold.



In physical systems, for instance, there is very often a distinction between the linear regime and non-linear regimes. In the linear regime the system is near a given rest state or EQUILIBRIUM state. Think of a spring that is gently pulled by some external force. For small force the spring's elongation is proportional to the force. If the force ceases to be, the spring will return to its rest shape. For larger forces this need not be the case. Now suppose that, during the formalization, we have derived formulas that are only valid in the linear regime, whereas, after the execution we found that a particular spring is elongated beyond this regime: in that case, the model outcome cannot be trusted[21].

In general: conclusions obtained in one regime cannot be carried over to another regime.

1.5.5 Reflection after the Conclusion Stage (5')

The initial problem was not stated in mathematical terms, and therefore the outcome of the execution of the formal model had to be translated back into non-mathematical terms. In this last reflection stage, we focus on the question: 'did we solve the initial problem', including 'did we do a proper presentation of the results and is our interpretation of the results adequate'? This reflection inevitably requires interviewing the problem owner: (s)he is the only one with enough contextual knowledge to assess if the initial problem was indeed solved.

We summarize the modeling process in Table 1.3.

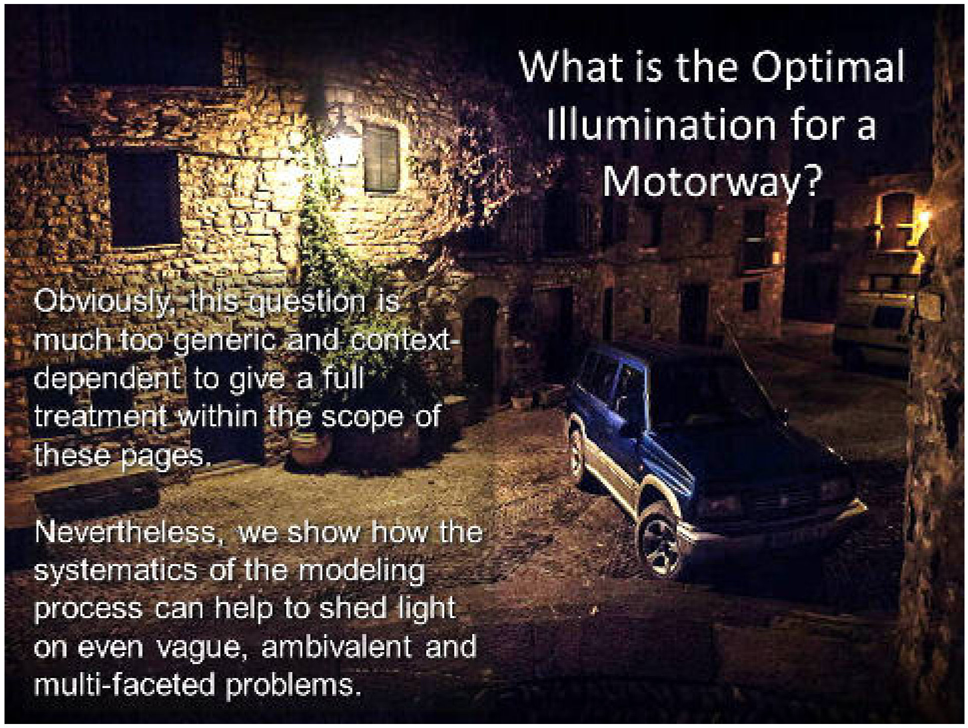
[21]The image of an equilibrist is taken from http://commons.wikimedia.org/wiki/Category:Balance#mediaviewer/File:UPSTREAM_FITNESS-5.jpg

Stage	What to do	Reflection
1. Definition	**(1.1) Formulate the Model's Purpose** • who is the problem owner? • who are the stake holders? • what is the problem context? • what purpose(s) do we have to deal with?	**(1') Assess the Plausibility of the Problem Definition** • does the problem owner recognize the redefined version of the problem? • under what conditions can the problem be considered to be solved?
2. Conceptualization	**(2.1) Identify Entities and Properties** • what are the most important entities and properties? • what are their relations?	**(2') Assess the Plausibility of the Conceptual Model** • do we include the crucial entities and properties? • do we include the crucial relations?
	(2.2) Identify Relations between Entities and Properties • which entities occur in relations? • which properties occur in relations? • what is the meaning of these relations?	• is the conceptual model sufficiently simple?
3. Formalization	**(3.1) Obtain Values for Quantities** • glass box: how do we obtain values for input quantities? • black box: do we have raw data or processed data? • do we understand the assumptions that underly these data?	**(3') Assess the Plausibility of the Formal Model** • is there ground truth data to validate the model? • can 'special cases' help test the model?
	(3.2) Formalize Relations between Quantities • what causal mechanisms should each relation express? • for every relation, what assumptions underlie this relation? • how to express the mechanism we want in mathematics, logic or computer language?	• are there independent models to test our model with?
4. Execution	**(4.1) Do Operations with the Model** • what sort of operations do we do? • how do we do these operations?	**(4') Assess the Plausibility of the Result** • do the results comply with assumptions? • are they valid for the purpose?
	(4.2) Obtain a result • in what form does the result arrive? • when do we have sufficient results?	• do we need to refine the model?
5. Conclusion	**(5.1) Present the Result** • what presentation styles exist for this type of result? • what presentation is adequate, given the problem owner? • does the presentation capture the essence of our result?	**(5') Assess the Plausibility of the Answer to the Initial Purpose** • has the initial purpose been fulfilled? • does the model outcome further contribute to solving the initial problem?
	(5.2) Interpret the Result • to whom should the interpretation be meaningful? • what does this person need to know? • is the interpretation valid? • does this interpretation raise any further questions?	
Evaluation		**Learn from what you have done** • what did go really well? • how can we consolidate this? • what did not go that well? • how can we improve this?

Table 1.3: Overview of the modeling process

1.6 Example

In this section we show how the modeling process as described in Section 1.4 could be executed in practice. We consider the case of illuminating a segment of public motorway with street lamps[22]. Notice: the various stages of the modeling process are explained in more detail in forthcoming chapters. In each stage we will come back to the streetlamp example to show how the introduced techniques apply there. The elaboration in the present section is therefore no more than a first, brief, introduction.



(1.1) Formulate the Model's Purpose

Some examples of purposes for a street lamp model could be:

- *verify:* could LED lamps be used for illuminating the road segment?
- *decide:* should we yes or no have adaptive illumination?
- *optimize:* what is the best height for lanterns for this road segment?
- *steer or control:* what is the strategy for switching the adaptive illumination on/off in real-time?
- *analyze:* how do the benefits of adaptive illumination depend on the traffic flow?
- ... and many others.

We choose as initial purpose 'optimization': in terms of costs, what would be the optimal way to illuminate the road?

(2.1) Identify Entities and Properties

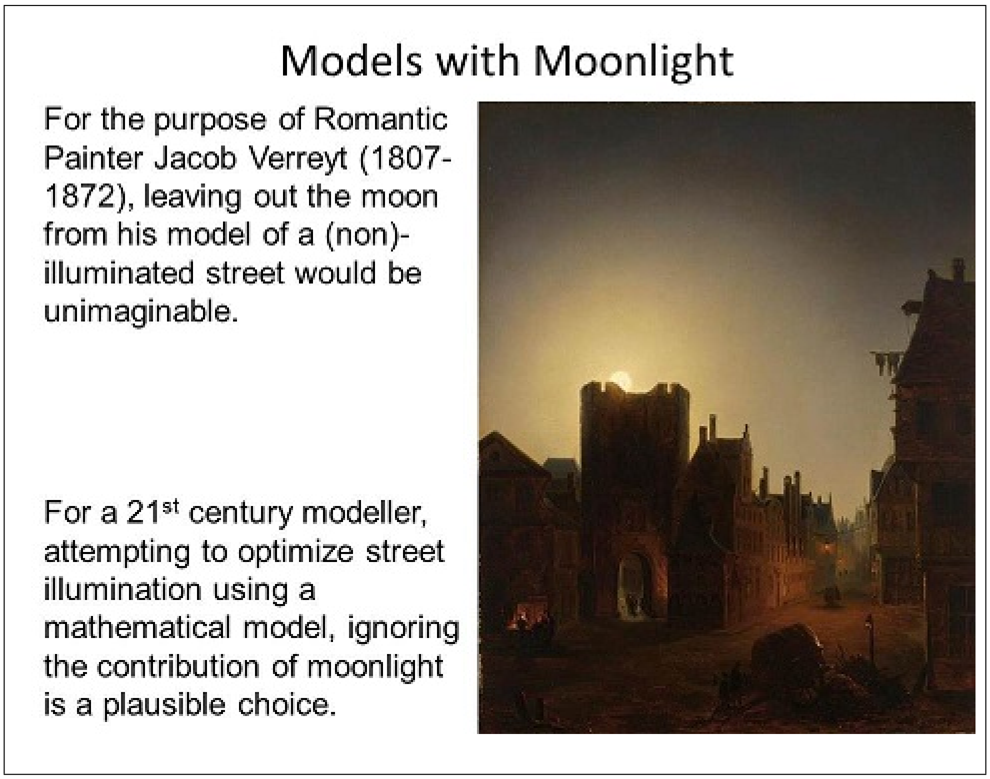
The problem owner is the provincial road agency; stake holders are car drivers, people living nearby (think of light pollution!), street workers, energy producers, insurance companies and perhaps many more.

In this stage we must decide what the SCOPE of the model will be: for instance, in case a detailed prediction of the long term financial exploitation is required, maintenance and replacement should be modeled as well. This scope gives us a first hint as to what entities should be considered, and

[22] The illustration is taken from http://commons.wikimedia.org/wiki/Category:Darkness#mediaviewer/File:4x4_(9351322825).jpg

therefore: which concepts should be present in the model.

The crucial question that should be answered is: 'with how little money can we safely illuminate the motorway?', and perhaps: 'could we obtain additional savings if street lamps are temporarily switched off during the absence of cars?' Entities involved therefore include motor traffic, drivers, street lamps, energy, and perhaps the equipment to implement adaptivity.



Next we look at the properties of these concepts that should be accounted for[23].

- *motor traffic* : The concept 'motor traffic' is complex. In order to answer the purpose for the model, however, we can capture the essential features in few properties, for instance the average time between the passage of two subsequent cars, and perhaps the average speed of the cars.
- *drivers* : The concept 'drivers' has a number of properties. Some have a physical origin (what is the minimum light intensity needed to see a road marking at some given distance?; what is the maximum light intensity to prevent blinding?), others come from psychological mechanisms (what is the minimal distance a driver needs to see an illuminated road segment in front of him in order to drive safely?).
- *street lamps* : The concept 'street lamps' can be captured by the distance between subsequent lamps, their height, and their illumination power.

A more detailed elaboration is given in Section 2.5.

(2.2) Identify Relations between Entities and Properties

Relations among concepts should be made explicit. E.g., the intensity of the street lamps has a relation to the energy that is consumed, and the energy has a relation to the costs of the illumination system. Further, there is a relation between the intensity of the street lamps and the reflection on the road as perceived by the drivers - and this perceived illumination level has a relation to the maximum and minimum values we should reckon with.

(2') Assess the Plausibility of the Conceptual Model

In Chapter 2 we learn that the conceptual model is drawn in the form of a graph. Then there are

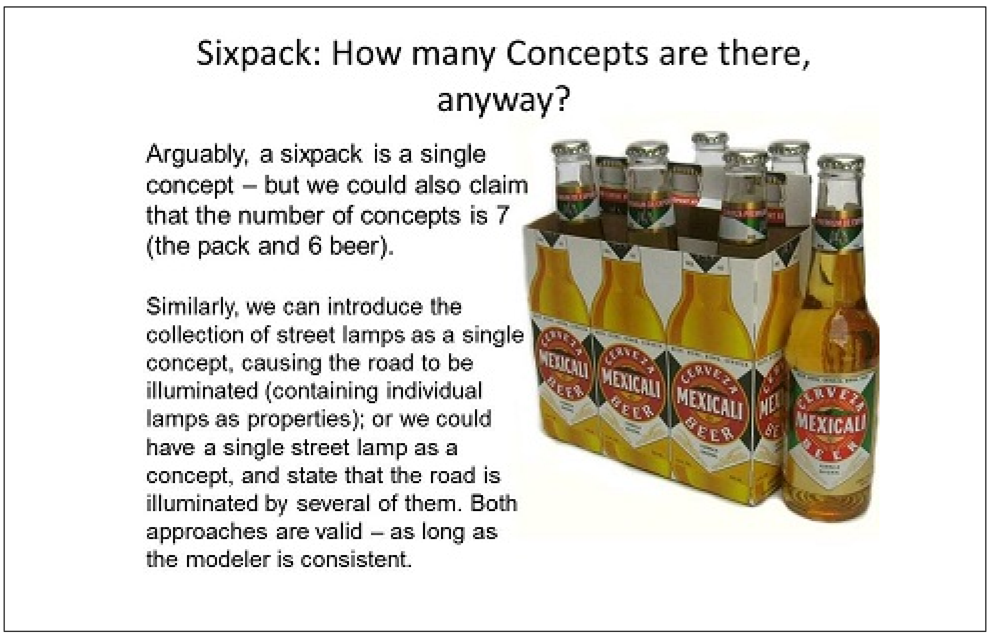
[23] The reproduction of the Jacob Verreyt painting was taken from http://commons.wikimedia.org/wiki/File:Moonlit_streetscene-Jacob_Verreyt.jpg

some early SANITY CHECKS we could perform. For instance, concepts or properties that don't engage in any relation should be taken out of the conceptual model. Also, if there are insufficient concepts or properties to express the initial purpose, the conceptual model cannot be complete.

(3.1) Obtain Values for Quantities

For the various quantities, there are different sources for their values, and hence different procedures to obtain them. Some examples:

- *motor traffic, drivers* : properties related to traffic[24] and drivers may need data gathering and aggregation. To asses the opinions of drivers about the new lighting scheme, for instance, may require a market survey including an experimental set up using a driving simulator.
- *street lamps* : properties of the street lamps are, within bounds, free for a designer to choose, so these could be used to find an optimal configuration.



(3.2) Formalize Relations between Quantities

Relations occur in many different forms, such as:

- Quantities have dimensions (energy, distance, energy / distance, time, price / distance, etc.). As we will see in Section 2.7, studying these dimensions will help to construct the right mathematical relations.
- Some relations will have the form of dependencies: the light intensity on a given point at the surface of the road depends on the height, the distance and the power of the lanterns. In Chapter **??**, we will see how a careful, step-by-step derivation of such dependencies leads to a plausible and transparent formal model.
- Other relations are constraints: the minimum intensity in any point should not be less than a certain value, and similar for the maximum intensity. In Chapter **??** we will learn how constraints can be represented mathematically, and how they can be resolved in simple cases.

(3') Assess the Plausibility of the Formal Model

Some behaviors of the traffic - street lamps system are intuitive, and the model should reproduce these. For instance:

- if lamps are taller, the light distribution underneath will fluctuate less;
- with a given power per lantern, energy consumption should be proportional to the number of lanterns;

[24]The photograph of a six pack was taken from `http://commons.wikimedia.org/wiki/File:Mexicali_Beer_6_Pack.jpg`

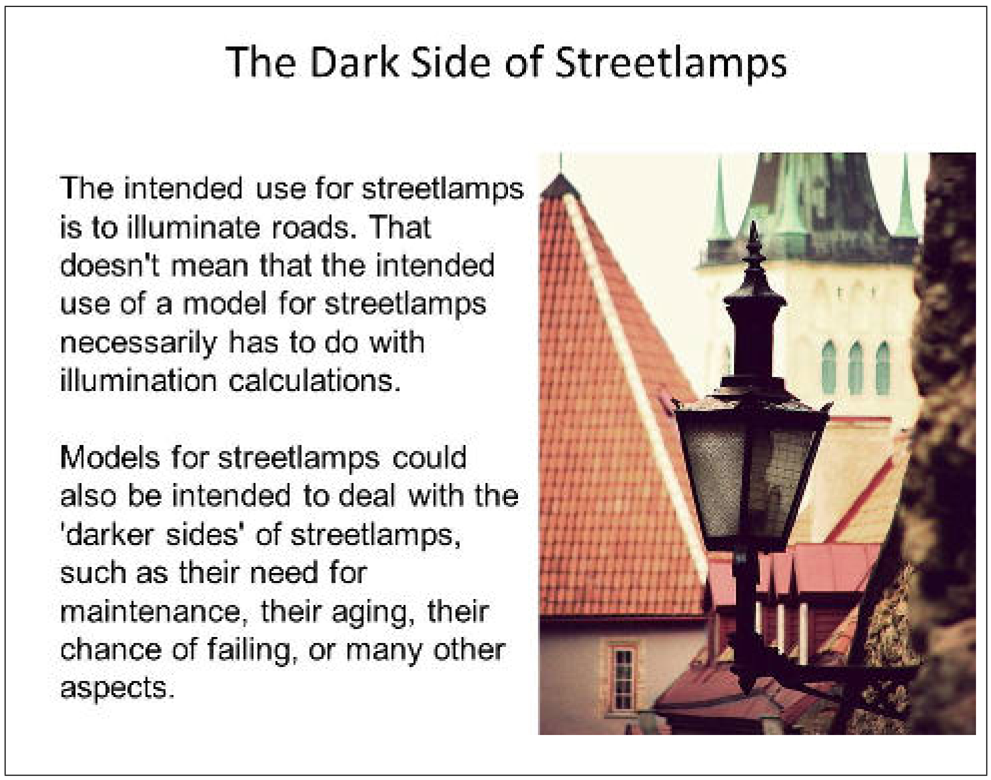
- local extrema in the illumination are likely to occur immediately below street lamps, and in the middle between adjacent street lamps (provided that these are modeled as point sources);
- if the lower limit for the intensity, needed for drivers to see road markings, decreases, energy consumption should decrease.

In general, assessing the plausibility of a formal model is called its *validation*; in Chapter **??** we learn which techniques can be used.

(4.1) Do Operations with the Model

Required operations depend on the purpose[25]. Some examples:

- *conditional prediction* : given a certain traffic density and given a configuration of street lamps, what will be the power consumption for adaptive street lighting;
- *optimization* : assuming non-adaptive illumination only, what will be the optimal configuration of street lamps (in terms of height, distance and power) such that energy costs are minimal;
- *optimization* : given a certain traffic density, what will be the optimal configuration of street lamps such that power consumption is minimal for adaptive street lighting;
- *analysis* : given a street lamp configuration and adaptive illumination scheme, how do energy savings depend on traffic distributions?
- *simulation* : given a street lamp configuration and a scheme for adaptive lighting, would drivers appreciate this scheme?



(4.2) Obtain a Result Depending on the operation, results could be:

- *conditional prediction* : an average amount of power consumption;
- *optimization* (for non-adaptive schemes): height, distance and power per street lamp for an optimal configuration;
- *optimization* (for adaptive schemes, for a given average traffic density): height, distance and power per street lamp for an optimal configuration;

[25] The photograph of a worn-out streetlamp in Talinn was taken from http://commons.wikimedia.org/wiki/File:Talinn-Aare_Piiroja_-Vintage_series.jpg?uselang=nl

- *analysis* : a table listing the energy savings for a number of average traffic densities (cars/hour);
- *simulation* : a collection of data, obtained from interviewing drivers that have made a test run in the simulator, regarding their opinions on adaptive lighting.

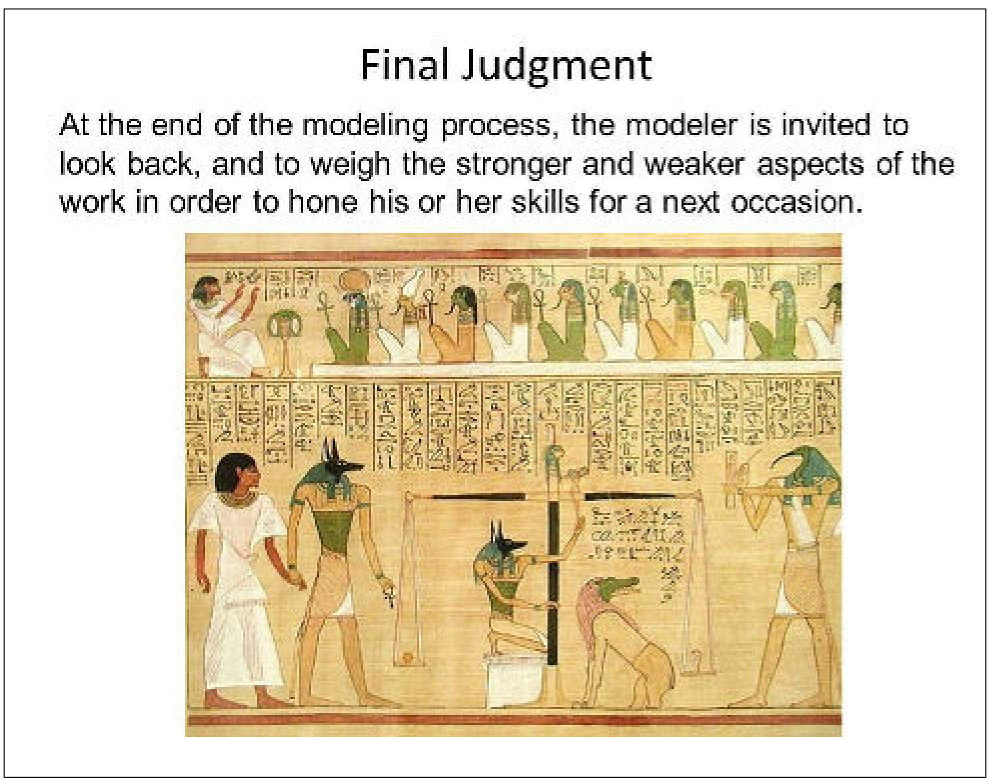
(4') Assess the Plausibility of the Result

As an example we consider optimization: the power consumption should be calculated for values of height, distance between street lamps and power per lamp for a range of values near the optimal values to verify if the optimum is *stable*. That is: if a small change in one of the quantities would cause a dramatic change in the calculated power consumption, we should mistrust the relevance of the outcome (see further Chapter **??**).

(5.1) Present the Result

In order to convince the problem owner and other stake holders, a mere presentation of the data (height, distance, power) is certainly insufficient. A presentation might consist of graphs showing the energy consumption as a function of each of these quantities.

(5.2) Interpret the Result



An interpretation should attempt to account for (some of) the approximations and assumptions used in the model. It should also try to provide some intuition behind the model outcome: 'in hindsight, the result could be understood because ...'. For instance, it might try to explain why the optimal height of the street lamps in an adaptive lighting scenario is less than that in a standard, non adaptive lighting scenario.

(5') Assess the Plausibility of the Answer to the Initial Problem

The only way to do this is by actual conversation with the original problem owner.

Learn from what you have done [26]

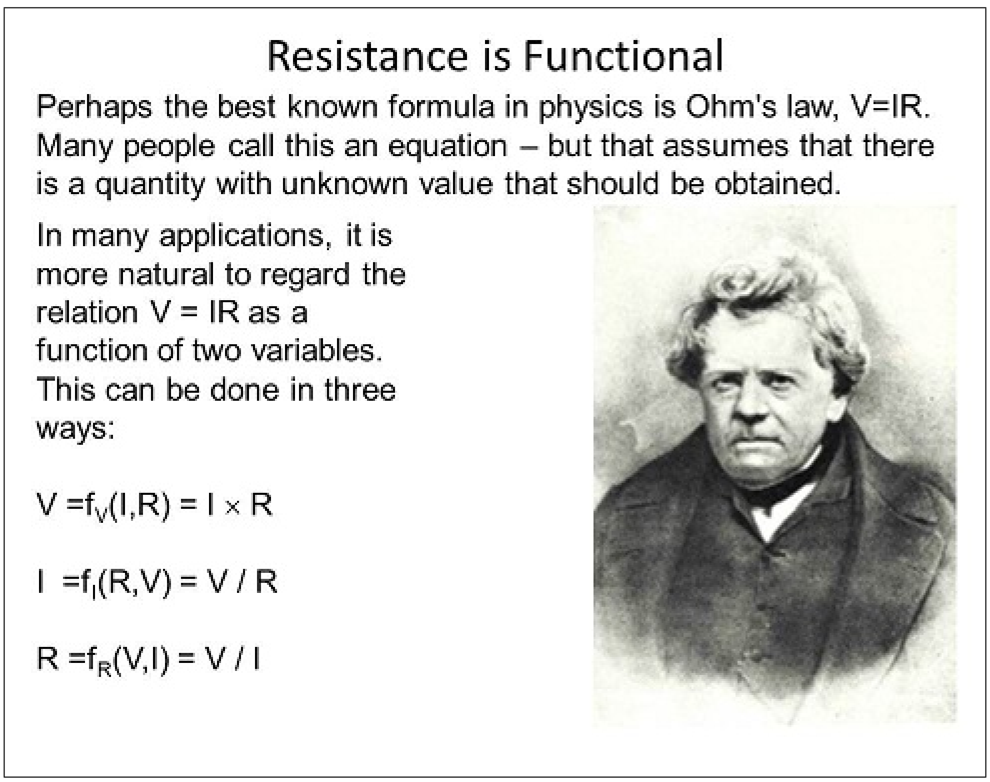
A straightforward approach to modeling the illumination distribution over a road surface due to street lamps could take both width and length of the road into account, that is, consider it a

[26] The image of the Egyptian, jackal-headed god Anubis, weighting the heart of the deceased (individual clad in white, left) and compare its weight to that of a feather to come to a final assessment 'sufficient' or 'insufficient', was taken from http://commons.wikimedia.org/wiki/Category:Papyrus_of_Hunefer#mediaviewer/File:BD_Hunefer_cropped_1.jpg

2D problem. Perhaps this is not necessary, though. Maybe the variation of the light intensity perpendicular to the road axis can be ignored in comparison to the axial direction. That is: we could interpret the outcome of the model as ground truth data, and approximate these with a simpler 1-D model. If the accuracy of the approximation is sufficient, we might have done our optimization calculations much more efficiently. This is good to know for a next problem involving road illumination.

1.7 Mathematical Tools: Functions of Two Variables

In this chapter models have been discussed. Models in which, amongst other things, quantities play a role. Moreover, relations between quantities have to be defined in the modeling process. Many relations are expressed as functions. Sometimes one has to maximize or minimize a function. For all the above things we use the analysis of real functions. Elementary calculus, however, does not elaborate on functions of several variables. Yet, in many models functional relations involving multiple quantities play a role.



An example is the problem, introduced in Figure 'Shining Light on a Narrow Passage' in Section 1.2.2, where the length of the red and green period of traffic lights has to be set. The function that gives the average waiting time depends on multiple quantities, for three of which a value has to be chosen. Therefore we introduce the essentials of functions of several variables here.

1.7.1 Functions of Several Variables

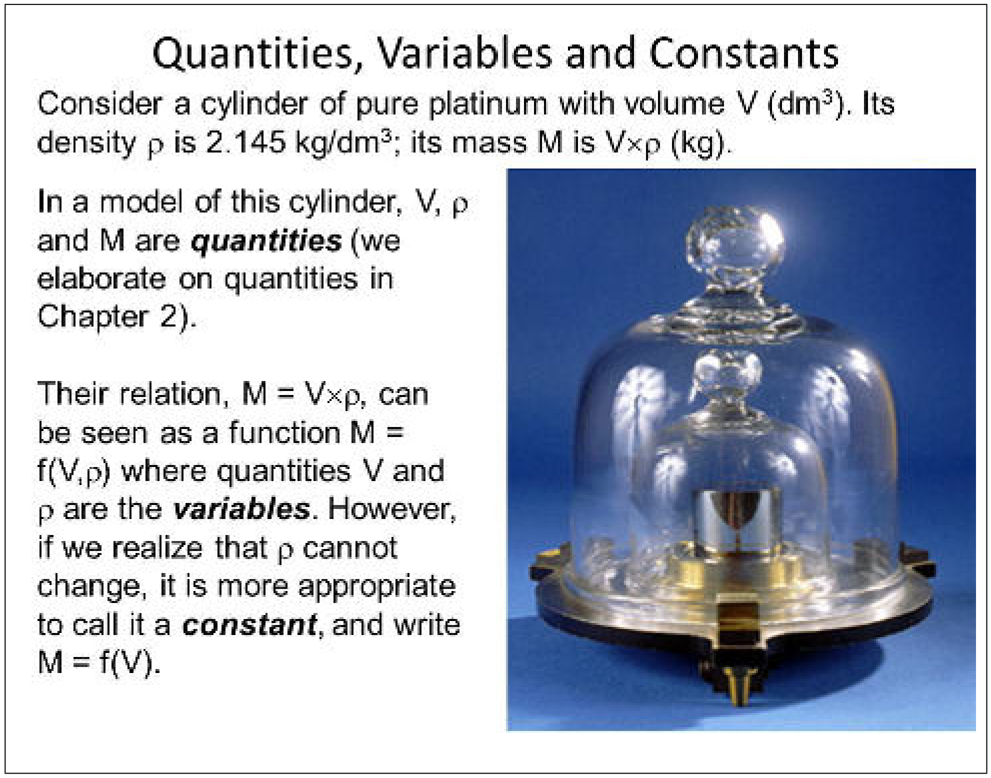
In the core course Calculus functions $f(x)$ of one variable x have been discussed. The domain and range of these functions were subsets of the real numbers. Now we need functions of several variables[27]. First we discuss functions of two variables.

A FUNCTION OF TWO VARIABLES is a rule that assigns a real number $f(x, y)$ to each ordered

[27] The portrait of Georg (not: Gerog!) Ohm is taken from http://commons.wikimedia.org/wiki/File:Gerog_Ohm.jpg?uselang=nl

pair of real numbers (x, y) in the domain $D \subset \mathbb{R}^2$ of the function. As for functions of one variable, the DOMAIN CONVENTION is that the domain of a function of two variables is the largest set of pairs (x, y) for which the function $f(x, y)$ can be evaluated, unless the domain is explicitly given by a smaller set, due to constraints coming from the modeled system. Analogously one can define a function of n variables.

Example 1: The function $f(x, y) = \sqrt{x^2 + y^2}$ gives the distance of a point to the origin.



Example 2: In Section 1.2.2 we mentioned the traffic lights problem. Here the length of the red and green period of traffic lights have to be set. The following pattern for the traffic lights can be chosen:

direction I	$R_0\ G_1\ R_0\ R_1$
direction II	$R_0\ R_2\ R_0\ G_2$

A purpose might be to minimize the average waiting time for a car. One needs a model for the traffic flow to estimate the average waiting time for a car. Suppose that f_i is the flow of traffic in direction i in number of cars per minute $(i = 1, 2)$ and f_0 is the number of cars per minute that can pass the part of the road with one lane. Furthermore, let R_0 is the time both traffic lights are red simultaneously and R_i is the time the traffic light in direction i is red and the other traffic light is green $(i = 1, 2;\ R_1 = G_2; R_2 = G_1)$.

For the model we assume that the cars arrive in a deterministic pattern with a constant inter arrival time (see also the discussion in Section 1.2.2). Furthermore we assume that the variables above have such values that there is no queue anymore at the moment that the traffic light turns red again. It can be shown that in this case the average waiting time F is

$$F = \frac{f_0}{2(f_1 + f_2)} \cdot \frac{\frac{f_1}{f_0 - f_1}(2R_0 + R_1)^2 + \frac{f_2}{f_0 - f_2}(2R_0 + R_2)^2}{2R_0 + R_1 + R_2}. \tag{1.2}$$

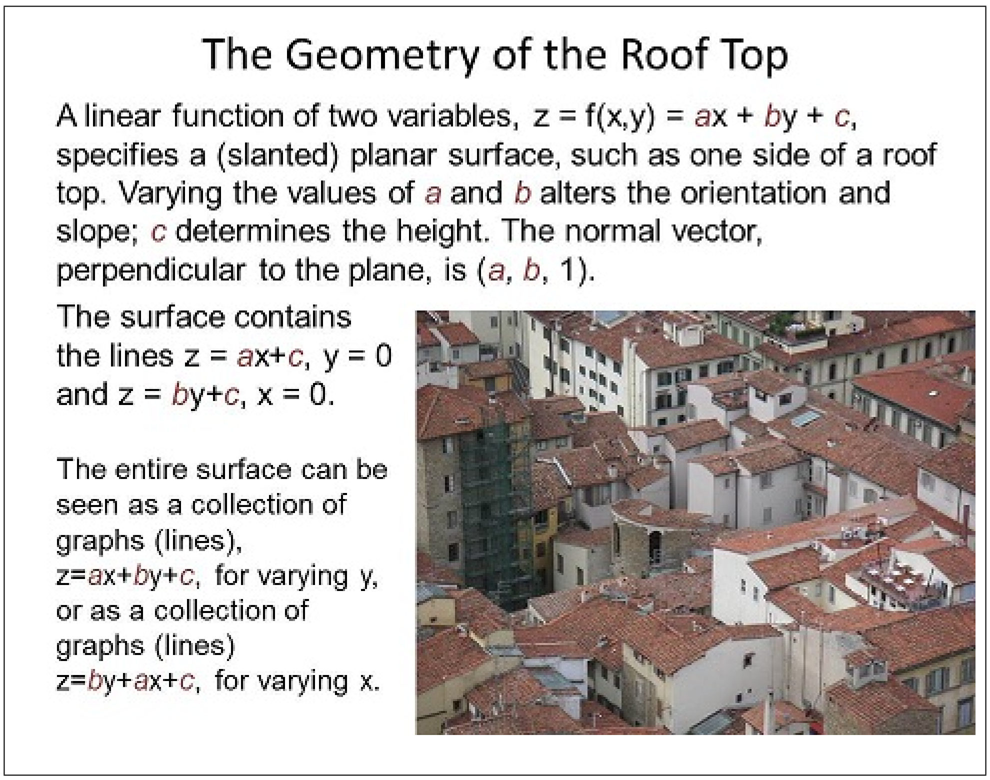
The derivation of this formula is given in Appendix A.

This function can be seen as a function of 6 VARIABLES[28]. The frequencies f_0, f_1 and f_2, however, are given or can be measured for the given road. The values of these quantities won't

[28]The photograph of the standard kilogram was taken from http://commons.wikimedia.org/wiki/Category:Kilogram#mediaviewer/File:National_prototype_kilogram_K20_replica.jpg

change; f_0, f_1 and f_2 are considered to be CONSTANTS. As follows: suppose that for the specific road 800 cars arrive per hour in one direction and 300 cars in the other direction. So, $f_1 = 40/3$ cars per minute and $f_2 = 5$ cars per minute. Furthermore, we assume that 40 cars can pass the blocked part of the road in one minute in one direction if the traffic light is green ($f_0 = 40$ cars per minute). With f_0, f_1, and f_2 being constant, F has been reduced to a function of three variables (R_0, R_1 and R_2, in minutes). If we further suppose that the length of the blocked part of the road is 500 m and the speed of the cars in the road work zone is 20 km/hour, then R_0 should be at least 1.5 minutes. If we choose $R_0 = 1.5$ (=another constant), then F reduces further to a function of only two variables. The average waiting time (in minutes) is then equal to

$$F = \frac{6}{77} \cdot \frac{7(3 + R_1)^2 + 2(3 + R_2)^2}{3 + R_1 + R_2}. \tag{1.3}$$



The function of Expression 1.3 is a function of two variables (R_1 and R_2). This function should be minimized. If one deals with a function of one variable, the derivative (if possible) is taken in order to find extremal values. However, now we have two variables. To find extremal values of functions of several variables one needs so-called *partial derivatives*. These will be introduced in Section 1.7.3.

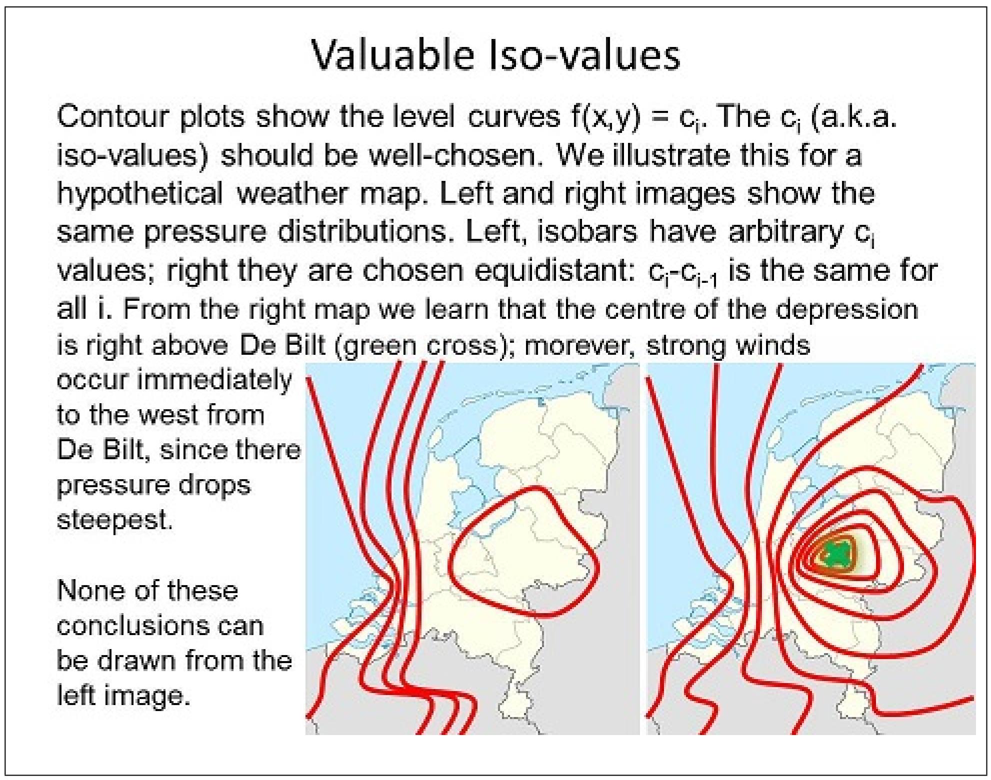
1.7.2 Graphs and Contour Plots

We are familiar with drawing a graph of a function of one variable. As we know this is a two-dimensional figure. Drawing the graph of a function of two variables results in a three-dimensional construction. Using smart projections one can visualize such a construction as an (two-dimensional) image. In the elementary course Calculus we have seen this for some simple functions. A plane[29] in three-dimensional space can be seen as a function of two variables. Consider for example the function $f(x, y) = (-4x - 5y + 32)/6$. If we write $z = (-4x - 5y + 32)/6$, then we recognize the plane $4x + 5y + 6z = 32$ with normal vector $(4, 5, 6)$. Here we use () to denote vectors. In Section 2.3 notational issues are discussed further. Go to this link to experiment with graphs for linear functions of two variables.

[29]The fotograph of rooftops was taken from http://www.rgbstock.nl/photo/n7LahDA/Rooftops+5

A graph of the function $f(x, y) = \sqrt{x^2 + y^2}$ of Example 1 in Section 1.7.1 can be found at this link ▷[26] A graph of the function given in Expression 1.2 of Example 2 in Section 1.7.1 is found at this link ▷[27] ; the values of the constants f_0, f_1, f_2, and R_0 can be adjusted at will. One can imagine that the visualization of a graph of a function of more than two variables becomes more or less impossible.

Another way to visualize a function of two variables in a two-dimensional picture is a CONTOUR PLOT[30]. A contour plot is a collection of level curves. A LEVEL CURVE of the function $f(x, y)$ is the two-dimensional curve defined by $f(x, y) = c$, for some constant c. The level curves of the function $f(x, y) = \sqrt{x^2 + y^2}$ are circles. If we would add a constant, say z_{shift} to f, the level curves would still be circles, although for different values of c. By going to this link, one can experiment by varying values for c, in combination with values for z_{shift}. Notice that, for any given function f, level curves don't necessarily exist for all values of c.



The contour plot of the the function F of Expression 1.2 can be found by going to this link. Here, the values for the constants f_0, f_1, f_2, and R_0 can again be adjusted.

1.7.3 Partial Derivatives

Recall that for a function $f(x)$ of one variable the *derivative function* $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

for values of x for which the limit exists. At a point $x = a$ the value $f'(a)$ is the instantaneous rate of change of the function $f(x)$ with respect to x.

For a function of two variables, *partial derivatives* can be defined.
The PARTIAL DERIVATIVE $\frac{\partial f}{\partial x}$ of $f(x, y)$ with respect to x is defined by

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \tag{1.4}$$

for values of (x, y) for which the limit exists.

[30] The map of the Netherlands was taken from http://commons.wikimedia.org/wiki/File:Netherlands_location_map.svg?uselang=nl

The *partial derivative* $\frac{\partial f}{\partial y}$ *of* $f(x,y)$ *with respect to* y is defined by

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}, \tag{1.5}$$

for values of (x,y) for which the limit exists.
Several notations are used for the partial derivative with respect to x. We have

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}f(x,y) = D_x f(x,y) = f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}(x,y). \tag{1.6}$$

Partial Derivatives and Zig-Zag Climbing

A function of two variables can be seen as a mountain landscape, and vice versa. Partial derivatives help to understand why mountain tracks follow zig-zag routes.

Suppose we are in (x,y), z=f(x,y), along a mountain track. We should take a next step (α, β) in some direction. The amount we rise from (x,y) → (x+α,y+β) is Δf = α ∂f/∂x + β ∂f/∂y.
We search α and β such that Δf is limited, 0 ≤ Δf ≤ c, for some small constant c. The solution is usually a zig-zag route.

As for functions of one variable a partial derivative gives the instantaneous rate of change. The instantaneous rate of change in the direction of x at the point (a,b) is given by $\frac{\partial f}{\partial x}(a,b)$.
It is just as easy (or difficult)[31] to compute partial derivatives as to compute derivatives of functions of one variable. If one computes the partial derivative of the function $f(x,y)$ with respect to x, just compute the ordinary derivative with respect to x, while treating y as a constant.

Example 1: The partial derivative $\frac{\partial f}{\partial x}$ of the function $f(x,y)=\sqrt{x^2+y^2}$ is equal to $\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot 2x$.

Example 2: The partial derivative of the function given in Expression 1.3 with respect to R_1 is equal to

$$\frac{\partial F}{\partial R_1} = \frac{6}{77}\cdot\frac{14(3+R_1)(3+R_1+R_2)-7(3+R_1)^2-2(3+R_2)^2}{(3+R_1+R_2)^2}. \tag{1.7}$$

1.7.4 Tangent Planes

For a function of one variable the *tangent line* to the graph $y=f(x)$ at $x=a$ is given by the equation

$$f'(a)(x-a)-(y-f(a))=0. \tag{1.8}$$

[31] The photograph of a Lebanese mountain track was taken from http://commons.wikimedia.org/wiki/File:Zig-zag_lebanese_mountain_road.jpg?uselang=nl

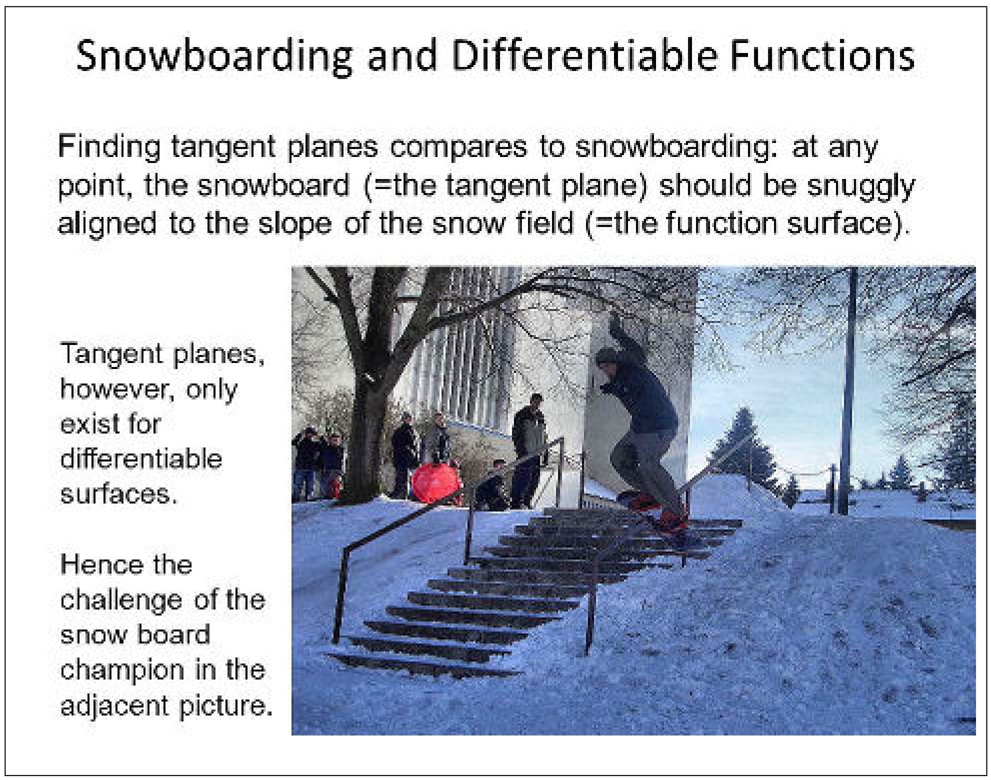
Using this the *linear approximation* of the function $f(x)$ can be defined. The linear approximation of $f(x)$ at $x = a$ is the function

$$L(x) = f'(a)(x - a) + f(a). \tag{1.9}$$

Likewise, there exists a linear approximation to a function of two variables[32] $f(x, y)$ at a point (a, b). Suppose that $f(x, y)$ has continuous first partial derivatives at (a, b). The equation of the TANGENT PLANE to the graph $z = f(x, y)$ at (a, b) is given by $f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$.

We conclude that a normal vector to the tangent plane is equal to $(f_x(a, b), f_y(a, b), -1)$.

By analogy with Expression 1.9, the LINEAR APPROXIMATION $L(x, y)$ of the function $f(x, y)$ at the point (a, b) is defined as $L(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$.



Example 1: The tangent plane to $z = f(x, y)$ with $f(x, y) = \sqrt{x^2 + y^2}$ at $(4, 3)$ is equal to $\frac{4}{5}(x - 4) + \frac{3}{5}(y - 3) - (z - 5) = 0$.

This can be written as $4x + 3y - 5z = 0$.

Example 2: The tangent plane to $z = F(R_1, R_2)$, where F is the function of Expression 1.3 at the point $(-\frac{5}{3}, \frac{5}{3})$ is equal to $z = \frac{16}{11}$. Note that this is a horizontal plane. Both partial derivatives are equal to zero.

To experiment with tangent planes and normal vectors, click here.

From now on, when we encounter a relation between 3 quantities, where one quantity depends on the other two, we can describe this as a function of two variables, and we have seen ways to visualise the functional dependency. In Section 2.8 we learn how we can do more sophisticated things with functions of multiple variables, such as finding extremes, which is useful for optimization models.

[32] The photograph of a snowboard trick is taken from http://commons.wikimedia.org/wiki/Snowboard#mediaviewer/File:Backside_Boardslide.jpg

1.8 Summary

- A model can only be meaningful with a clearly defined *purpose*;
  - purposes come from *research* (aim: to produce knowledge or understanding) or *design* (aim: to create or add value)
  - purposes are: *explanation*, *prediction* (unconditional, conditional), *compression*, *abstraction*, *unification*, *communication*, *analysis*, *verification*, *exploration*, *decision*, *optimization*, *specification*, *steering and control*, *training*. See Table 1.1.
- Modeling approaches can be distinguished on a number of *dimensions*:
  - *material - immaterial*: does the model involve material objects?
  - *static - dynamic*: does time play a role?
  - *continuous - discrete*: does the modeled system involve 'counting' or 'measuring'?
  - *numerical - symbolic*: do results follow from operations on numbers or on expressions?
  - *geometric - non-geometric*: do features from 2D or 3D space play a role?
  - *deterministic - stochastic*: does probability play a role?
  - *calculating - reasoning*: does the purpose rely on numbers or on propositions?
  - *black box - glass box*: does the model start from data or from mechanisms?
- Modeling is a process involving 5 stages, each stage consisting of one or two activities (=subsequent blocks in Table 1.3) and a reflection:
  - *definition*: establish the purpose
  - *conceptualization*: devise a representation of the modeled system in terms of concepts, properties and relations
  - *formalization*: devise a representation of the conceptual relations in terms of mathematical expressions
  - *execution*: perform the appropriate operations (often involves running a computer program)
  - *conclusion*: devise an adequate presentation and interpretation
- Functions of several variables;
  - A *function of two variables* $f(x, y)$ is a rule that assigns a real number $f(x, y)$ to each ordered pair of real numbers (x, y) in the domain $D \subset \mathbb{R}^2$ of the function
  - A graph of a function of two variables results in a three dimensional figure
  - A *level curve* of the function $f(x, y)$ is the two-dimensional curve defined by $f(x, y) = c$, for some constant c
  - A *contour plot* is the collection of a number of level curves
  - The *partial derivative* of the function $f(x, y)$ with respect to x is the ordinary derivative with respect to x, while treating y as a constant; the partial derivative with respect to y is the ordinary derivative with respect to y, while treating x as a constant

- The equation of the *tangent plane* to $z = f(x, y)$ at (a, b) can be found by use of the partial derivatives
- The *linear approximation* $L(x, y)$ of the function $f(x, y)$ at the point (a, b) can be found by use of the partial derivatives

1.9 Learning goals

1.9.1 Knowledge

You should be able to name at least 6 purposes for models; for all purposes, introduced in Section 1.2, you should know their differences and main characteristics. You should be able to name all dimensions introduced in Section 1.3. You should be able to name all stages in the modeling process as introduced in Section 1.4, and for each stage, you should know which activities of the modeling process take place in that stage. You should comprehend the meaning of all the terms introduced in this chapter, as they appear in the index. You should possess a working knowledge of the analysis of real functions of several variables as explained in Section 1.7 and of the relevant sections in the calculus book of either *Adams* or *Smith & Minton*:

Adams: §12.1 until 'Using maple graphics', §12.3 until 'Distance from a point to a surface', §12.6 until Definition 5 (so only Linear Approximations).

Smith & Minton: §12.1 until 'Density plots', §12.3 until 'Higher-order partial derivatives', §12.4 until 'Increments and differentials'.

1.9.2 Skills

In this section, with 'problem' we mean: a problem that does not require domain-specific knowledge exceeding your present knowledge.

For a model, developed in the context of a given problem, given with sufficient detail, you should be able to determine its purpose(s). For a given problem domain, you should be able to identify several possible purposes that models could have. For a model, given with sufficient detail, you should be able to determine each of its dimensions as introduced in Section 1.3. For a given problem domain with a given model with given purpose, you should be able to suggest a global direction for a modification to the model to satisfy an alternative purpose. For a given problem, you should be able to set up a proposal for an approach according to the modeling process.

For a given function of two variables, you should be able to find the domain and range, to draw graphs and contour plots and be able to interpret the plots. You should be able to compute the derivatives of a function of two variables, compute the tangent plane and the linear approximation of the function at a given point.

1.9.3 Attitude

When confronted with a problem that might benefit from a formal approach, you should consider to use a model. When approaching a problem by using a model, you should have the attitude to first formulate a purpose. When devising a model, you should consider the various modeling dimensions from Section 1.3 before you choose a definitive route. When approaching a problem by means of a model, you should be inclined to follow the modeling process as explained in Section

1.4. When dealing with quantitative dependencies among quantities, you should consider applying the calculus of functions of several variables.

1.10 Questions

1. Without using a model, how do people predict the weather? Are you sure that there is no model involved?
2. In your own words, explain the difference between the various types of predictions we discuss.
3. In your own words, explain the difference between purposes 'specification' and 'communication'.
4. In purpose 'steering or control' we talk about 'a human in the loop'. What does 'loop' mean here?
5. Explain in your own words the difference between purposes 'abstraction' and 'unification'.
6. Explain in your own words the difference between purposes 'decision' and 'optimization'.
7. Explain in your own words what 'continuity' means.
8. In your own words, give the meaning of 'sampling'.
9. What is the relation between 'sampling' and 'reconstruction'.
10. In your own words, explain what an angle is.
11. We say that 'angle is a special kind of distance'. What do we mean by that?
12. In your own words, what is 'emergent'?
13. We say 'logical expressions [...] resemble arithmetic expressions'. What do we mean by that?
14. In your own words, explain the difference between black box and glass box.
15. What is the difference between conceptualization and formalization?
16. Reflection on a modeling stage asks for plausibility, not for correctness. Why?
17. Give some similarities and differences between 'assumption' and 'hypothesis'.
18. What are the benefits of a log-log scale?
19. Both in step 3.1 (obtaining values for quantities) and in step 4.2 (obtaining a result) in Table 1.3, we obtain values. What is the difference between these steps?
20. In the example of the modeling proces in Section 1.6, why should it be that for a given traffic distribution, savings will be less if street lamps are further apart?
21. In the example of the modeling proces in Section 1.6, why could it be that the optimal height of the street lamps in an adaptive lighting scenario is different from that of a standard, non adaptive lighting scenario?
22. Why is the lower left most cell in Table 1.3 empty?

1.11 Exercises

1. Think of an example, analogous to the 'can-we-afford-to-buy-this-book-advisor model' where a prediction on the basis of mathematics at first sight seems fully reliable; next analyse under what assumptions the model holds, and analyse a scenario where the model could still go wrong.

2. (*) Consider Keppler's work based on Tycho Brahe's results, and Newton's work based on Keppler's work. In both cases you could say that X's model explains Y's results.

   (a) For (X,Y) being (Keppler, Brahe) and (Newton, Keppler), formulate what explains what;

   (b) Give an argument why these two forms of explanation are similar;

   (c) Give an argument in what respect the two forms of explanation differ.

3. We discussed 5 models for the solar system to illustrate various purposes for models, used in scientific research. Give another example of a system for which you give at least 3 very different models, and discuss the purposes of these models.

4. We claim that the purpose 'explanation' is to a large extent a social construct: it depends on the willingness of some community whether or not an explanation is acceptable. Give some examples of explanations of phenomena that are acceptable in one community, but not acceptable in another community.

5. Give an example where an explanation can be objectively shown to be wrong.

6. Give an example of a modeling situation with purpose 'explanation' without the possibility of 'prediction', and one with 'prediction' without the possibility of 'explanation'.

7. We discussed various purposes for models; Table 1.1 gives a summary. Not all purposes are independent, in the sense that fulfilling one purpose sometimes needs an other purpose to be fulfilled as well. Find examples of pairs of purposes where one purpose implies another purpose.

8. We explain linear *inter*polation in the text.

   (a) What is linear *extra*polation? (If necessary, look up the answer - but (re-)formulate it in your own terms).

   (b) Analogous to Expression 1.1, give a formula for linear extrapolation.

   (c) Think of an example where extrapolation is the answer to a modeling purpose.

   (d) Generally, extrapolation is thought to be riskier than interpolation. Give a reason.

9. The discussion of the black box model for birds mass and longevity did not start from a problem. Try to think of a problem where the subsequent models (inspiration model: plotting a graph; compression model: fitting the data points with a formula; prediction model: using the formula to say something about the expected longevity of a new bird species with given mass) could help to provide a solution.

10. Consider the example with bird's data from Section 1.3.8.

(a) Suppose that there are two sub species of blackbirds that accidentally were not distinguished when compiling the table with average masses and average longevities. What could be a worst case consequence for the model?

(b) Draw a general conclusion from this example with respect to the validity of averaging.

11. In 1.4.5, we state that a weather model where the output consists of numbers is unsuitable for the problem owner, for the stakeholders, and for the problem context. Give examples for each of the three, and explain why a purely numerical output is insufficient in these cases.

12. We make an inventory of some 15 purposes in Table 1.1. Consider one domain (examples of 'domain' in this context are the weather, the national economy, public transport, a treatment for some disease, transport phenomena, ...), and give for each purpose a plausible problem. You are not supposed to *solve* these problems, of course!

13. Apart from the purposes from Table 1.1, we discuss 7 dimensions that can help distinguishing modeling approaches. All in all this gives $2^7 \times 15 = 1920$ combinations. Find some (at least 3) examples of combinations of modeling purpose and modeling approach that are unlikely, and explain why these are unlikely.

14. In assignment 13, we propose 1920 combinations of modeling purposes and modeling approaches. Answer the following questions:

    (a) Pick one combination and give a casus (=a domain and a problem in this domain) for which this combination is fitting. The casus should be so concrete that we can verify that indeed this combination of purpose and each of the dimensions applies.

    (b) Change one of the dimensions and do the same.

    (c) Change the purpose and do the same.

15. In secondary school (in Holland: 'VWO') final exams physics, there are often assignments where 'something' has to be calculated. Find two of such assignments in different domains of physics (domains in physics are, for instance, mechanics, light, atomic physics, heat, electricity and magnetism, ...); for each answer the following questions:

    (a) Assume that this assignment is a model to help solving a problem. Propose a problem for which this assignment could be (part of) the model.

    (b) Using the purposes in Table 1.1, identify the purpose of this model.

    (c) Using the dimensions, characterise the approach.

16. Our modeling process contains both a conceptualization, a formalization and an execution stage. Think of a problem that is approached by means of a model, containing a conceptual model, where the formalization stage is skipped.

17. In Section 1.4.5 we give an example of an interpretation. Find another example yourself.

Exercises concerning mathematical tools (Section 1.7)

Adams: §12.1: 4, 6, 12, 14, 20, 23, 27, 28; §12.3: 2, 7, 13, 16; §12.6: 1, 3.

Smith and Minton: §12.1: 3, 4, 7(a), 7(b), 16, 27, 53, 54; §12.3: 1, 3, 19; §12.4: 1, 3, 7, 8.

Chapter 2

The Art of Omitting

'Making models is like playing golf: the holes cause the excitement'

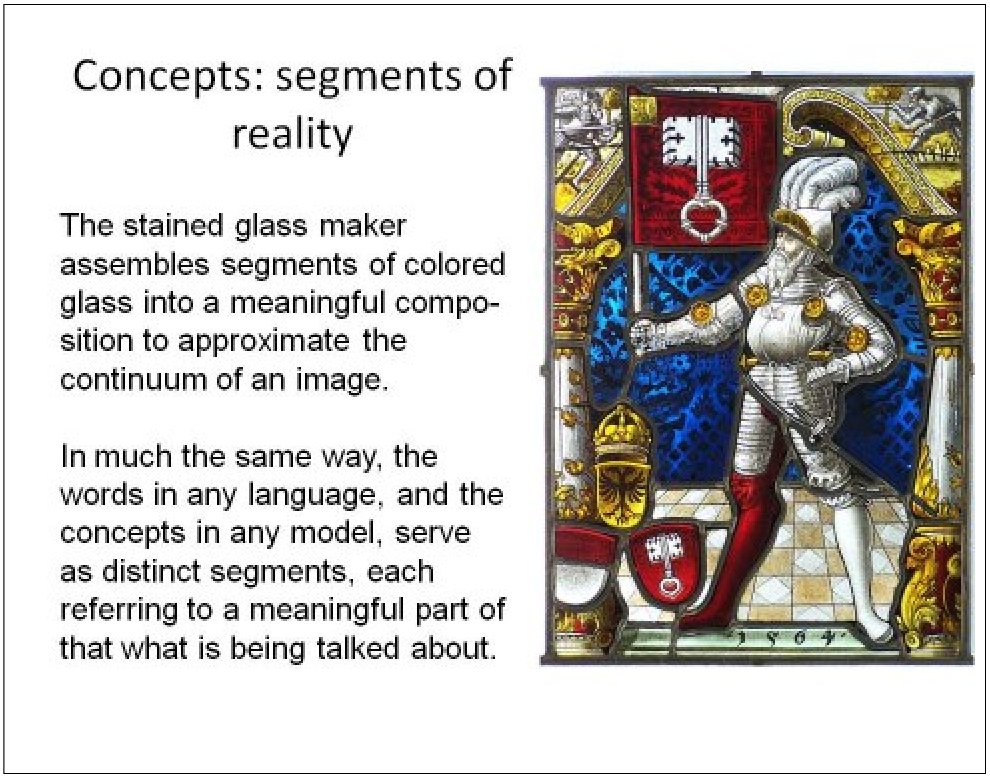
When the Spanish troops, headed by notorious conquistador Herman Cortez, invaded Middle America in the 16th century, natives were shocked. Not only did they suffer devastating losses because of the military superiority of the invaders, they also had to overcome their fear, being confronted with mythical creatures from hell: horrendous monsters, galloping on four legs and using two further extremities to shoot arrows. Had they only realized that these creatures were not single entities, but rather human soldiers on horseback, as some found out after a horseman fell from his saddle, their panic might have been less. They were not to blame ▷[28], of course: horse does not occur as an indigenous animal in America. What is really interesting, is that the Indians failed to see rider and animal as two separate entities.

2.1 The Conceptual Model

The ability to see entities as separate and DISJOINT, to some extent, is inborn. Young children, before the age of language, focuss their attention to distinguish parts of the environment, taking subsequent samples rather than having their gaze move in a smooth and featureless fashion. One second here, the next second there. This is a first form of SEGMENTATION. When words are learned, this takes on a next stage. Words invite to see the world as consisting of isolated objects. 'Nose' refers to a different part of the face than 'mouth'. These distinctions are meaningful: a nose has different purpose than a mouth (you should not put soup in your nose), and words help to differentiate. There are no separate words for the left part of the nose and the right part of the nose, which makes sense as long as there is no practical purpose for such discrimination. Apparently, language categorizes the things in the world into chunks in a meaningful way.

2.2 Concepts and Entities

Modelers create their own world. A world that is initially empty, and that becomes inhabited by the CONCEPTS [▷29], introduced, one by one, by the modeler.



The word 'concept'[1] derives from 'to conceive', that is: 'to imagine', 'to form as a mental image or as a thought'. A concept is a placeholder for an ENTITY in the system that is being modeled.

Concepts point to, or *refer* to entities in the modeled system. Once a concept enters the model, it receives a name. Names are important [▷30]: a name endows an entity with INDIVIDUAL IDENTITY. By means of a name, it can be distinguished from all other concepts [▷31]. A modeler should give conscious thought to naming every new concept added to the model.

Entities are only relevant for a modeler if they CORRESPOND to a concept in the model. Such concept is said to REPRESENT the entity it corresponds to. A concept represents the entity it was introduced for, in much the same way as a flag represents a country, or a letter in a Western alphabet represents a vocal sound.

Naming a concept not necessarily requires deep thinking. We may borrow names from everyday language. The concept that is to represent a lantern will be called `lantern`, rather than `X` or `Pineapple` [▷32]. Concept's names resemble the answer given to a child learning language when it asks 'how do you call *that*?', while pointing at something.

The `lantern`-concept as it occurs in a model allows the modeler to reason about lanterns, but it *is* not a lantern: never will it shine, and no dog will ever urinate against it. Most likely it will be a little rectangle on a piece of paper, together with other rectangles called `traffic`, `road`, etc., forming the conceptual model of the road illumination system.

[1] The image of a stained glass window was taken from http://commons.wikimedia.org/wiki/File:Muzeum_Su%C5%82kowskich_-_Zabytkowy_Witra%C5%BC.jpg?uselang=nl

2.3 Properties

An entity in the real world may surprise us: 'I didn't know that this lantern was rusty' or 'I did not even think of the possibility that it *could be* rusty'. In the first case we realized that 'rustiness' is a property[2] of lanterns, but we we hadn't assessed if this particular lantern was or was not rusty. In the second case, we did not even consider the property 'rustiness', until we saw brown spots, and realized that these were patches of rust.

Such a discovery is impossible for concepts in a conceptual model. All the properties of a concept are explicitly known, as they result from a definition by the modeler. Their value can also be the result of a definition, or it can result from calculating or inferring, using relations with other properties.

But: what, actually, *is* a property?

First, a PROPERTY is a means to distinguish concepts from each other. When segmenting the world into concepts, we need to say in what respect two concepts are different. In the example in the introduction of this chapter: one of the ways to distinguish horse and soldier is by the property `numberOfLegs`. A horse has four, and a soldier has two, and therefore they can be distinguished. Obviously there may be more distinguishing properties, for instance `hasTail`, `canTalk`, or `getsSalary`. For a horse the values of these properties are, respectively `true, false, false`, whereas for a soldier they are `false, true, true`. In all situations, when we have two concepts `C1` and `C2` that are different, there must be at least one property that takes on different values for `C1` and `C2`.

> **Hidden Behind a Closed Fence ...**
>
> A common meaning of 'property' is: 'that which is owned by somebody'. Often the word refers to real estate, such as a piece of land; the ownership may be clearly indicated by walls, fences and gates.
>
> 
>
> Our use of the word 'property' is much broader. We use the word 'property' to describe ***any*** attribute of a concept. Further, a property (e.g., `color`) is often shared among many concepts; its values (`green`, `red`, ...) help to distinguish among those.

It follows that a PROPERTY carries part of the information in a concept. It is an aspect of a concept. A concept's properties together carry all information in the concept.

A PROPERTY always comes in the form of a *name* and a *set of values*.

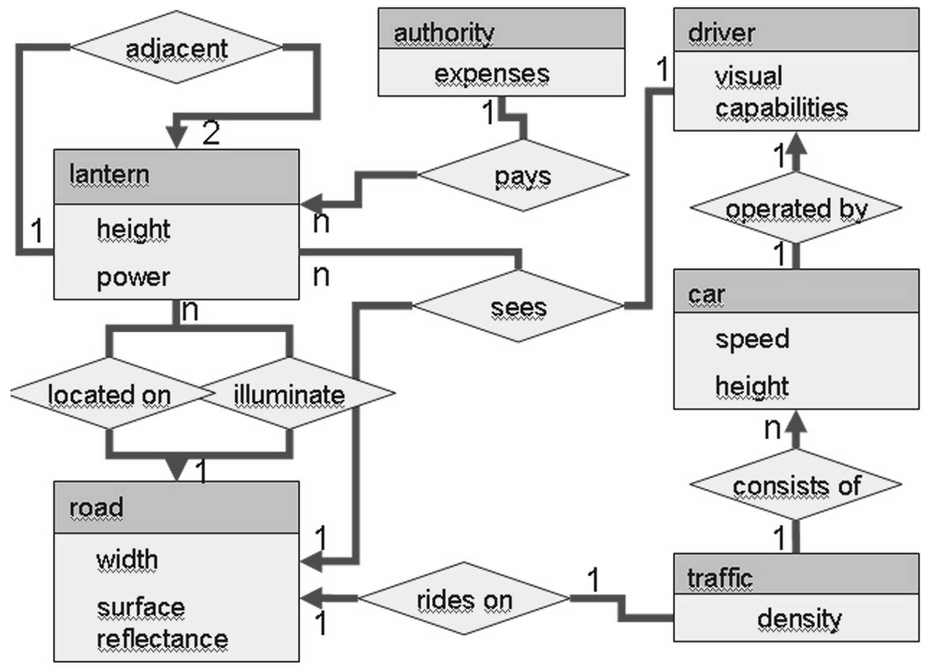
The name of a property is used to refer to the property. So we can talk about the property `color` with name `'color'`.

The set of values a property can have is called the TYPE of that property. The type of a property

[2] The image of a gate is taken from http://commons.wikimedia.org/wiki/File:K%C3%B6nigstetten_-_G%C3%B6ttweiger_Herrenhof,_Portal.JPG?uselang=nl

can be limited to just a single element. A set with one element is called a SINGLETON. If a property's type is a singleton, the element of the type is called 'the VALUE' of the property.

The property `color`, could, depending on the purpose of the model, have a singleton value such as {`green`}, or a set {`green`, `red`}, or a RANGE, {`light Green ... darkGreen`} - assuming that, for some color, we can assess if it is 'between' light green and dark green. We might also define the type `colors` to represent all colors that can be distinguished, e.g. by a human being or on a computer screen. In a *range* all elements are known when only two extreme values and a notion of ordering is given. For instance, the *range* of all integers between 3 and 6 is the *set* {3,4,5,6}; the ordering is '<': $3 < 4$, $4 < 5$, $5 < 6$, and 4 and 5 are the only two integers x with $3 < x < 6$. Ranges are denoted with three dots between the lower and upper element; so {`lightGreen ... darkGreen`} is indeed a range of colors.

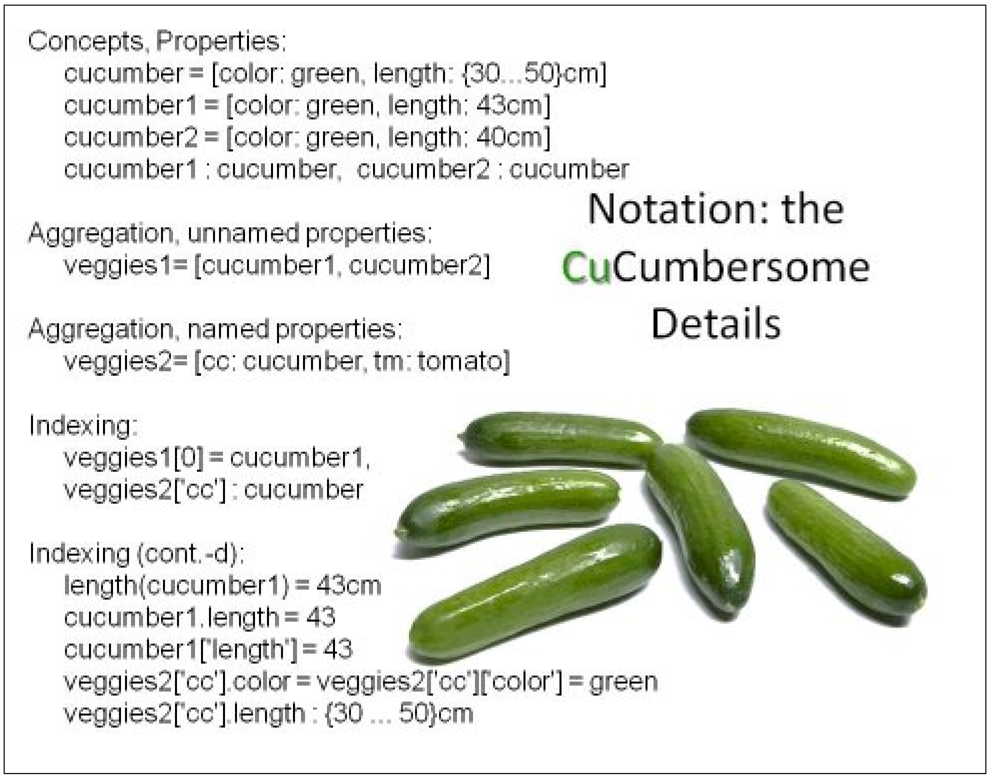


Conceptual model in the form of an Entity Relation Graph

Figure 2.1: Part of a model for the road illumination problem. Concepts are denoted as rectangular blocks. Every concept has a name (top) and perhaps some properties (below the name). Relations are directed arcs, or arrows. Relations have names, written in the diamond shape label. The arity of a relation is indicated at both ends of each arc. Most relations connect two concepts. The relation 'sees', however, connects three concepts: indeed, 'seeing' involves (multiple) lanterns, a driver and a road. (Illustration source: Kees van Overveld)

In summary: a property is a chunk of information about a concept. All information about a concept is captured in its properties. We say: a CONCEPT is defined as a bundle of properties.

An example of the idea of a 'bundle of properties' is a vector. In mathematics we write e.g. $(4, 2, -3)$ or $\langle 4, 2, -3\rangle$ or $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$, all meaning that the numbers 4, 2 and -3 are not to be considered as loose values, but instead belong together - for example because they refer to a location in 3D space. We say that the values 4, 2, -3 are AGGREGATED. The differences in notations occur because the development of mathematical notation took place over several centuries with contributions of numerous authors. This is not problematic, as long as mathematical expressions are only used by humans, and as long as notation in a single document is consistent. Since some decades, however, mathematical notation is also used to program computers, and mathematical expressions are to be communicated over the Internet. For those goals the need of standardization is more urgent. So if we denote aggregations (such as vectors), to be used in the context of automated processing, we need to do so in a uniform manner. Our vector then becomes `[4,2,-3]`.



In this example[3] of an aggregation, we don't give explicit names for the properties. Sometimes this is acceptable. By convention, when denoting a 3D location by a vector, the first number denotes the horizontal coordinate, the second number the vertical coordinate, and the third number is the depth. In computer context, we can refer to one of the members of an aggregation without named properties by setting `p[0]` to refer to the first element of vector `p`; `p[1]` to refer to the second element, et cetera. The number between [and], used to single out one element from an aggregation, is called the INDEX.

In many cases, however, we want property names to be given explicitly. Then we write, again for the same vector, `[x:4,y:2,z:-3]`. Here, `x` is the name of a property with value 4, etc. Using named properties, we don't have to be careful with the order of the elements: `[x:4,y:2,z:-3]` =`[z:-3,x:4,y:2]`, whereas `[4,2,-3]`$\neq$`[-3,4,2]`. If an aggregation is given with named properties, we refer to the value of a property with name `x` as `p['x']` (notice that we need quotes here to signify that `x` is the name of a property, and not an index. In the expression `[x:4,y:2,z:-3]` quotes surrounding `x`, `y`, or `z` are not necessary, because the only thing that can occur before ':' is a name: the name of a concept or the name of a property).

Our notation allows values to be numbers, but other types are also admitted. We may even write down an aggregation such as `M=[[1,0,0],[0,1,0],[0,0,1]]`. This is a vector with three (unnamed) elements, being `[1,0,0]`, `[0,1,0]`, and `[0,0,1]`. In other words, the elements of `M` are themselves vectors, and `M` is a vector of vectors - in other words, a matrix. In a mathematical context we would write `M` as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

We use square brackets, [and], to denote aggregation instead of parentheses, (and). Apart from convention, this has also a conceptual reason. Paretheses are used to denote FUNCTION

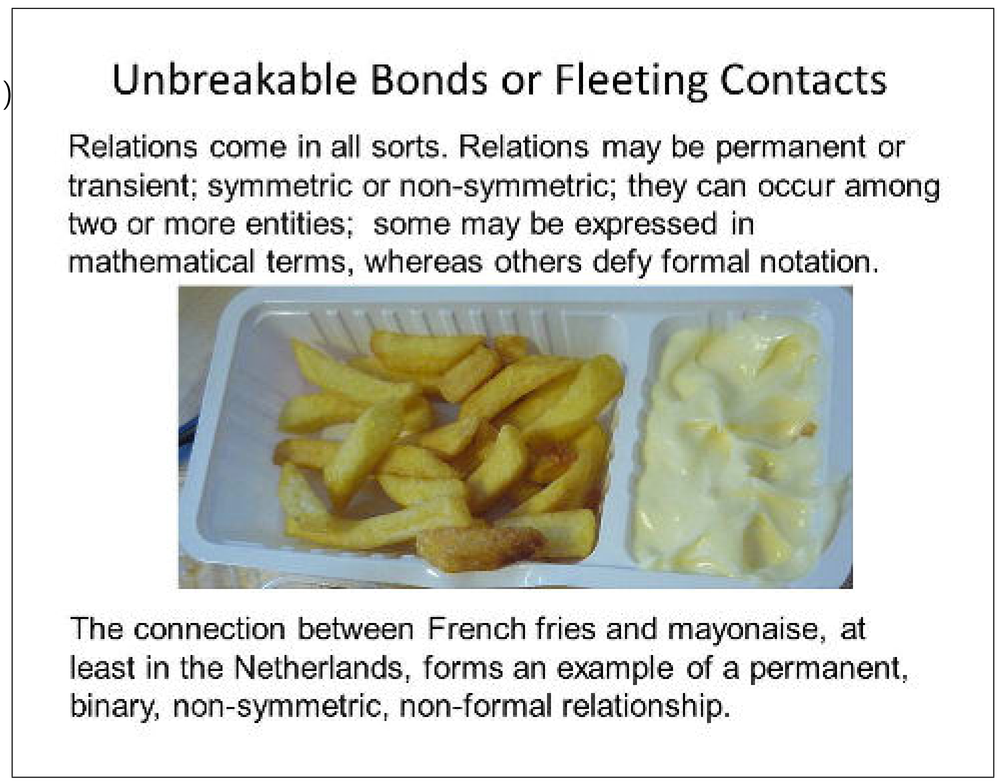
[3]The image of cucumbers was taken from http://www.rgbstock.com/photo/mqyBYfE/cucumber

application. For a function $f : x \to x^2$, the expression $f(3)$ denotes the function f, applied to the argument value 3, yielding 9.

Functions don't necessarily need to be mathematical calculations (as in the example: taking the square). Any recipe that associates, to an element from a set called this recipe's DOMAIN, a single unique element from a set called this recipe's RANGE, is a function. For instance, the property `color` takes as domain the set of all concepts that have a color. `tomato` and `canary` are part of the domain of `color`; e.g. `water`, `wind`, and `democracy` are not. Therefore, `color` can be called a function, and we can formally write `color(tomato)=red` (notice the use of parentheses instead of square brackets). Here, `tomato` is the ARGUMENT and `red` is the RETURN VALUE of `color`. Although many computer languages may not support this, we can view *any* property `P` of a concept `C` as a function, where `C` is an element of its domain, and `P(C)` is a value in its range. If `C` is an aggregation with named properties, we could just as well write `C['P']` for `P(C)` [▷33] .

2.4 Relations

In natural language[4], we distinguish substantives ('nose', 'lantern', 'dog'), verbs ('sneeze', 'shine', 'bark') and PREPOSITIONS. Prepositions *connect* substantives. They often correspond to RELATIONS. In particular relations in space or *spatial relations*. 'The dog is near the lantern' could be replaced by `near(dog,lantern)` without much loss of information [▷34] .



Apart from prepositions, *verbs* also often connect substantives. The verb 'sees', above, was an example; other examples are: `produces(machine, sausages)` for 'this machine *produces* sausages', `wait(passenger,train)` for 'the passenger *waits for* the train', `likes(john,marshMellows)` for 'John *likes* marsh mellows', etc. For this reason, relations are also often expressed by verbs.

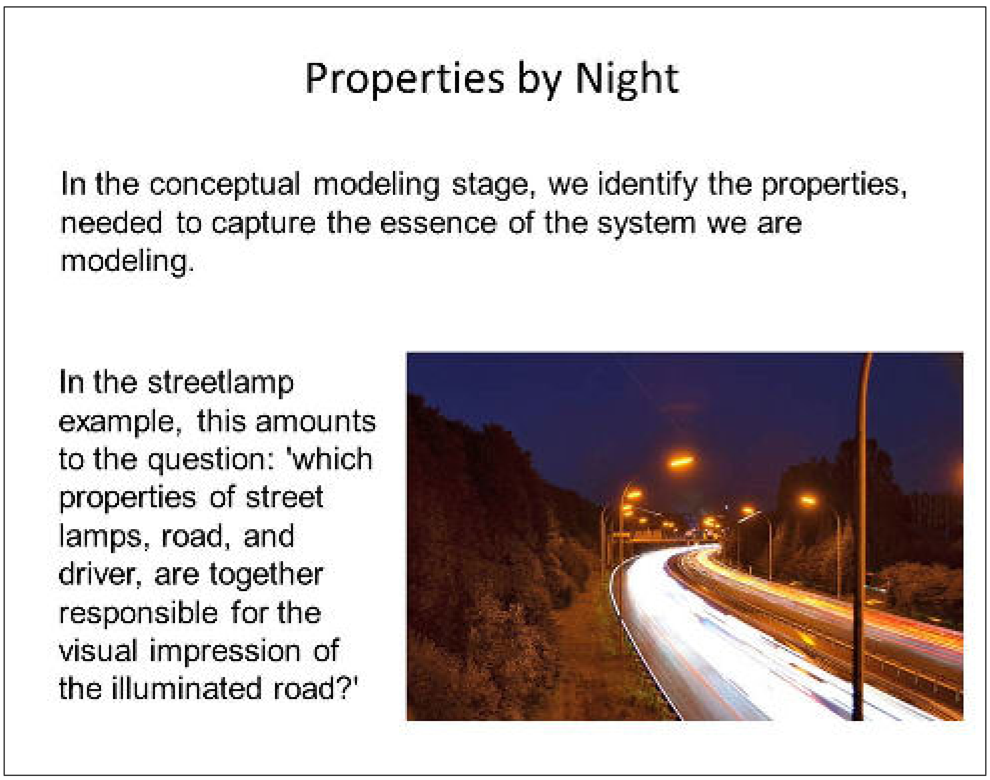
If there is more than one concept in our model, it is necessary to say something about the relations in the model (which concept is related to which concept?), and to say what these relations mean.

[4] The image of French fries comes from http://commons.wikimedia.org/wiki/File:Pommes_frites_med_fritessaus.jpg?uselang=nl

There is a significant difference between the meaning of 'dog near lantern' and 'dog above lantern', which again is different from 'lantern above dog'.

Just as properties, relations can also be seen as functions. In Section 1.7, we have seen that a function can have multiple arguments. An example of a relation, denoted as a function would be `smarter(John,Peter)`. This function yields the value `true` if `John` is smarter than `Peter`, and `false` otherwise. Its domain is the set of pairs of humans; its range is the set {`true`,`false`}. Again, as with properties, we see that a relation-seen-as-a-function does not necessarily mean that there is some mathematical computation involved: it could be that the outcome of the function evaluation `smarter(x,y)` amounts to looking up some information on `x` and `y`.

2.5 Constructing a Conceptual Model



Construction of a conceptual model[5] is described with a 4-step process. We follow the street lamp example from Section 1.6.

1: establish concepts. First identify the entities for which we need corresponding concepts. We write down things that come to mind when we think of road illumination. Say, `lantern`, `road`, `moon`, `car`, `tree` ▷35
.

After brief reflection, we want to skip `moon`, because our system should also work on moonless nights, and we skip `tree` because trees complicate things and should be omitted for a first iteration. Next we add two more concepts, `driver` and `traffic`, because without either of these, illuminating roads is pointless. A first inventory of concepts should at least contain enough concepts to be able to formulate the problem.

Often the main challenge of this step is not to include too many concepts. Conceptual models often are unnecessarily complex because they contain concepts contributing little to the purpose of the model, but obscuring its working. Hence the title of this chapter: the art of making a good conceptual model is the art of omitting the unnecessary.

[5] The image of an illuminated motorway was taken from http://commons.wikimedia.org/wiki/File:Motorway_(7858495690).jpg?uselang=nl

2: establish properties. For each of the concepts we ask 'what do we need to know of this concept?', in other words: 'which properties do we need?'. For the concepts found in the example in step 1, this may yield the following:

lantern: `height`, `power`;

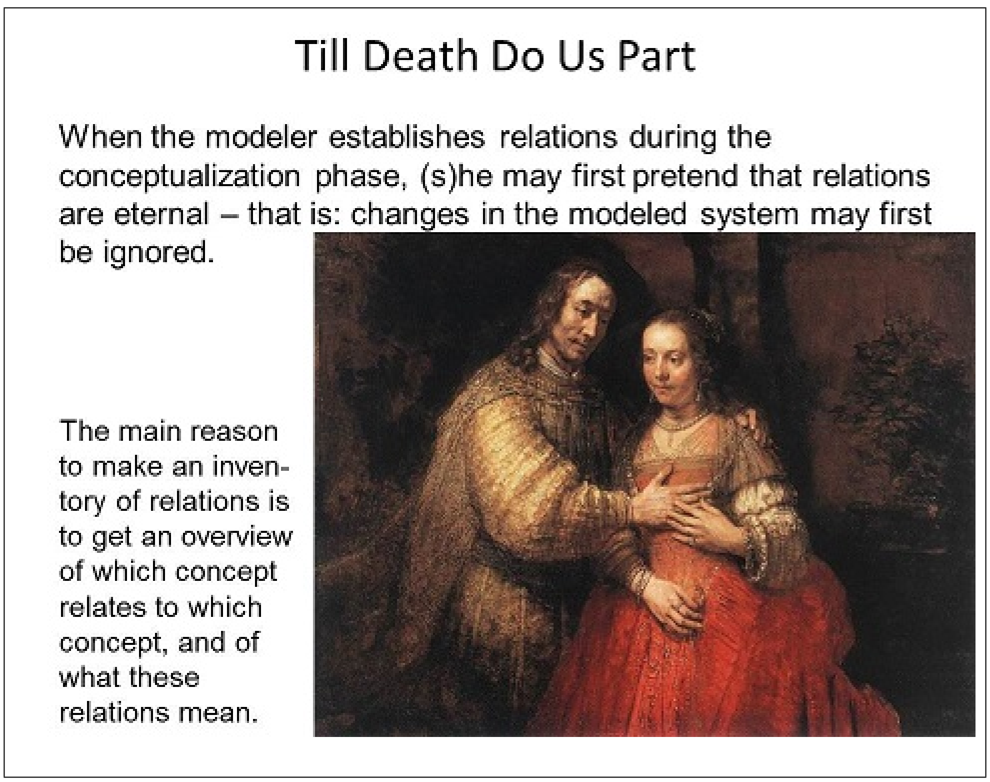
road: `width`, `reflectivity`;

car: `speed`, `height`;

driver: `visualCapabilities`;

traffic: `density`.

Some of these properties are clearly needed (like the height and power of the lantern). Others may be discarded after a moment of reflection. E.g, the height of the car: a truck driver sits 2 meters above the road surface, and a motorist perhaps not even 1 meter, but they experience the illumination conditions not very differently. Yet others may require additional work: the visual capabilities of a driver cannot be represented by just a number.



3: establish types of the properties. Every property has a type, determining its set of values[6]. Here, we think of values for the following properties (we use the so-called dot notation, to be explained in detail in Section **??**. The expression `a.b` means 'property `b` of concept `a`').

lantern.height: {5.0 ... 25.0}m - this is the range of heights for lamp posts; the actual value could correspond to an optimum sought for, in case the purpose of the model is 'optimization'. The unit 'm' (meter) in the notation for the type of `lantern.height` signifies that this property is an amount of meters.

lantern.power: {100, 2000}W - these are the powers of LED lamps and gas discharge lamps, respectively. This may come in if the purpose of the model is to aid in what-if analysis or decision support: in this case, the consequence of taking LED lamps or gas discharge lamps.

road.width: {14.40}m - this is a singleton, being the measured width of the segment of road

[6] The reproduction of 'The Jewish Bride' was taken from http://commons.wikimedia.org/wiki/File:Rembrandt_-_The_Jewish_Bride_-_WGA19158.jpg?uselang=nl

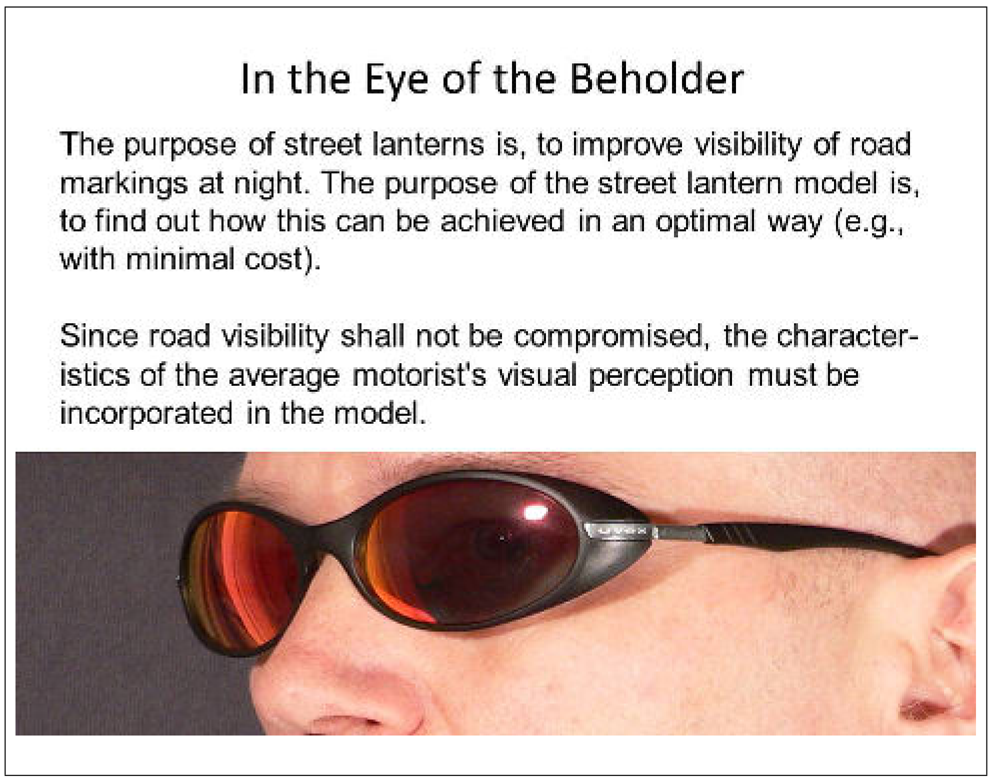
we need to illuminate. It is a constant in the model. When there is no risk for confusion, we may leave out the accolades in a singleton: instead of setting `road.width` $\in \{14.40\}$, we may set `road.width=14.40`. But if there is an uncertainty interval associated with a value, it is appropriate to write that `road.width` is an element of the set $\{14.30 \ldots 14.50\}$.

road.reflectivity: `reflectivity` - this means: we don't know yet the value or the value range for the reflectivity. It will follow from a separate model or from experiments. Therefore we denote it as a named type: `reflectivity`, which will be a singleton, or an uncertainty range if it results from a measurement.

car.height: $\{1.0 \ldots 3.0\}$m - this range of values may be used to check if the final solution is not sensitive to the actual height of the driver, as we supposed earlier.

traffic.speed: $\{20 \ldots 180\}$km/h - this indicates a range of speeds for which we should test the validity or applicability of the model. Does our model still make sense if cars go very fast?

driver.visualCapabilities: `driverView`.



Everything we need to know about drivers' visual capabilities[7] cannot be captured in a single number. These capabilities include the minimal luminance so that road marking can be distinguished, and the maximal luminance so that blinding does not occur. We need a *new concept* that contains the perceptual characteristics of the average driver. This concept, that still is to be detailed, is called `driverView`. The property `visualCapabilities` has a COMPOUND type: its value is a concept with properties, such as `minimalLuminance`, `maximalLuminance`, and perhaps others. The opposite of a compound type is an ATOMIC type. Numeric values, strings and booleans are examples of atomic types.

traffic.density: $\{30\}$cars/minute. The value of this property may result from aggregation. Perhaps measurements of the actual traffic over a period of time are available.

authority.expenses: `real`. In this step we realize that we had forgotten a concept in our model, namely `authority`, with property `expenses`. Without this property in our model, we could not express the purpose of finding an *optimal* solution. We don't know the range of

[7]The image of sunglasses is taken from http://commons.wikimedia.org/wiki/Glasses#mediaviewer/File:Sonnenbrille_fcm.jpg

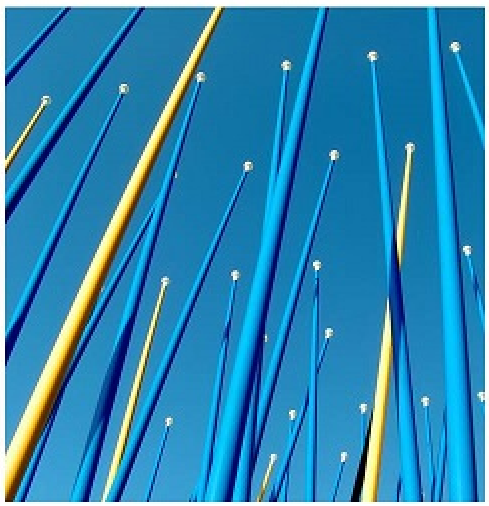
`authority.expenses` yet, hence the type `real` ▷36 . We want it to be as little as possible, though.

4: establish relations. [8]

Tall Taller Tallest

To a property, we associate a value. The type of a property is the set of values this property can assume.

While making the conceptual model, it is often possible immediately to assign a set of values to a property – even if eventually the property turns out to possess a single value. Setting `lantern.height = {5.0 ... 25.0}m` means that the height of a lantern will be somewhere between 5 and 25 meters, perhaps because a lantern manufacturer produces lanterns in these lengths.



Next we seek relations between concepts ▷37 . In principle, we could exhaustively check all pairs of concepts and ask 'is there a relation between these two concepts?' ▷38 . Below we give a list of relations that may emerge; other sets of relations may also be adequate:

- *illuminate(lantern(n), road(1))* - to express that the road is illuminated by multiple lanterns;
- *operatedBy(car(1), driver(1))* - to express that a car is operated by a driver. So the location of the driver will be fixed with respect to the location of the car;
- *consistsOf(traffic(1), car(n))* - to express that traffic is an aggregation of multiple cars;
- *ridesOn(car(n), road(1))* - to express that the location of any car is constrained to the road;
- *sees(driver(1), road (1), lantern(n))* - to express that the illumination, perceived by the driver, comes from light, emitted by lanterns, reflected on the road. Notice that this is a relation between 3 instead of 2 concepts;
- *pays(authority(1), lantern(n))* - to express that the costs of installing and operating the lanterns are to be paid by the authority responsible for lighting the motorways;
- *adjacent(lantern(1), lantern(2))* - to express that lanterns are adjacent to each other, in other words that each lantern has two adjacent lanterns ▷39 .
- *locatedOn(lantern(n), road(1))* - to express, for instance, if lanterns are located on the axis of the road, or at both sides, et cetera.

In this list, the numbers in brackets indicate the number of concepts involved in the relation. 'n' means '1 or more'. These so-called *arities* are further explained in Appendix **??**.

It usually requires several iterations before the lists of concepts, properties, values and relations are appropriate. At any time, we should check against the purpose of the eventual model.

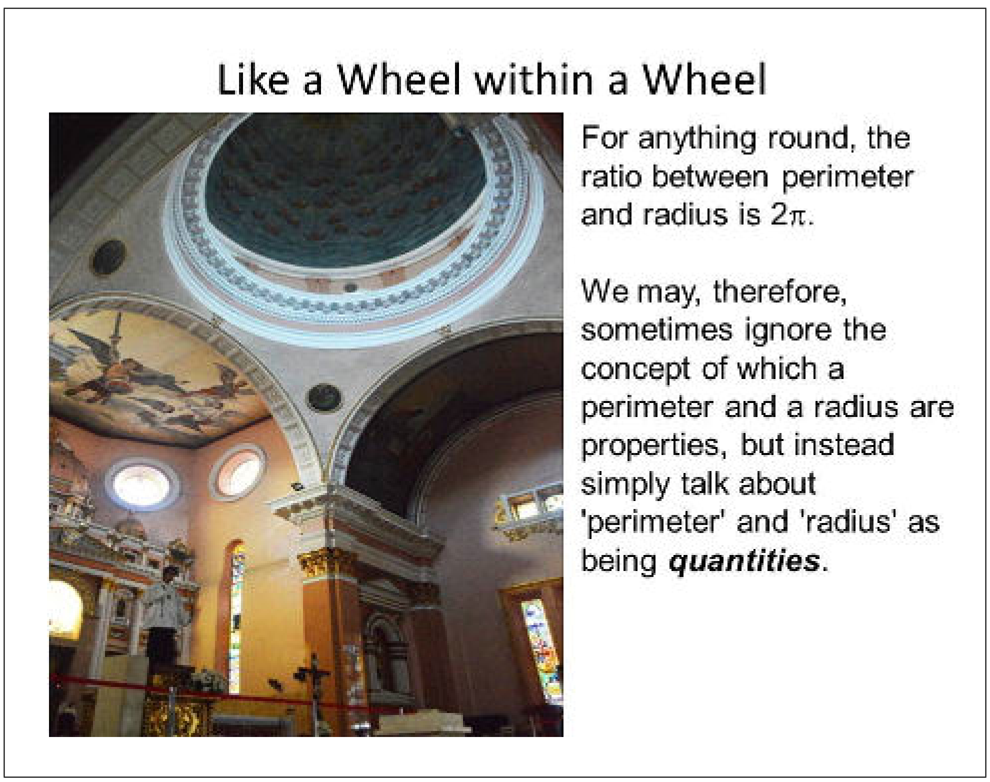
[8]The image of flagpoles is taken from http://commons.wikimedia.org/wiki/Category:Flagpoles#mediaviewer/File:Cloetta_Center,_Flagpoles.jpg

Using relations, a conceptual model can be graphically depicted as in Figure 2.1. Such a drawing is called an ENTITY-RELATION GRAPH. The nodes (usually drawn as boxes) in an entity-relation graph are the concepts from our conceptual model. Other terms for such graphical representations are SEMANTIC NETWORK or CONCEPT GRAPH.

Relations in an entity-relation graph need to be indicated by arrows, since all but symmetric relations have a direction: `R(A,B)` generally means something different from `R(B,A)`. Every node represents a concept and is depicted as a box; this box contains the name of the concept and more information, such as its properties and perhaps their types.

Concepts, properties, and value sets, although they form a natural perspective on the world, are quite subtle. In Appendix **??** we go somewhat deeper into some issues for dealing unambiguously with conceptual models.

2.6 Quantities



A property is always a property *of* some concept. We never encounter isolated properties. For some purposes, however, we don't need to know which concept some property is a property of[9]. For instance, to calculate the perimeter of a circle, it does not matter if this circle is the shape of a blood vessel, a piece of land or the lid of a bucket with paint. In all cases the same formula applies. Mathematicians commonly talk about QUANTITIES, disregarding the concept that the quantity is a property of. Knowing how to compute the perimeter of a circle can be applied to the concepts `bloodVessel`, `pieceOfLand` and `bucketWithPaint.lid` without further consideration of other properties ▷40 . Therefore it is adequate to talk about 'quantities'. We already know that every property has a type, therefore quantities have a type, too. The TYPE of a quantity is defined as the set of values the quantity can assume. Since we may not know what concept a quantity, such as `radius`, is a property of, the type of a quantity does not relate to something known of any concept. The

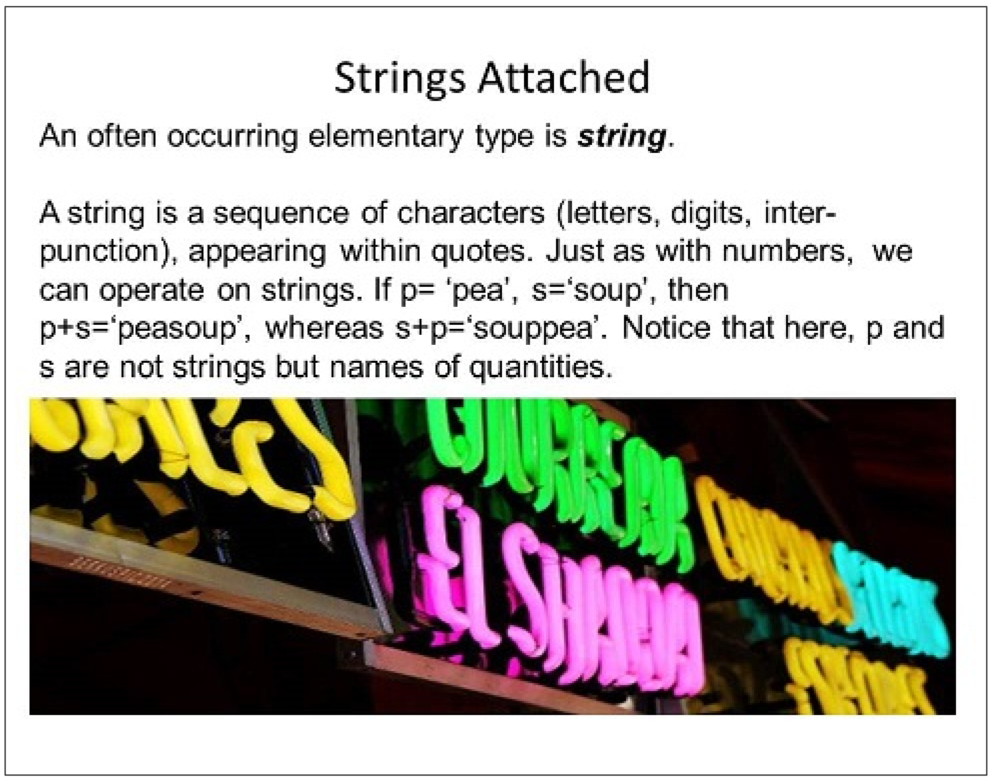
[9] The image of Binondo church is taken of http://upload.wikimedia.org/wikipedia/commons/1/15/Binondo_Church_Circular_Configuration.jpg?uselang=nl

type of a quantity, corresponding to property $C.P$ is therefore the union of all possible types of the property $C_i.P$ for all possible concepts C_i. In the case of `radius`, the type of the radius of a bicycle wheel is $\{0.3...1.8\}$m, but a radius in general can be any non-negative real number. Therefore the type of a quantity *radius* will be the non-negative reals.

To denote the difference between quantities that occur as properties of a concept versus quantities that appear without a conceptual context, we write the former in `this font`, whereas the second will be written in *this font*. So: "`myBicycle.frontWheel.perimeter= 6.28 * myBicycle.frontWheel.radius`" as opposed to "*perimeter*=2×π×*radius*".

We use the term 'quantity', where other texts would use words such as PARAMETER, VARIABLE, FACTOR, TERM, or COEFFICIENT.

The words 'parameter', 'variable', 'factor', 'term', or 'coefficient' all have slightly different meanings. These meanings vary over the disciplines; within one discipline the meaning can be different in different contexts. What is called 'coefficient' in one discipline might be 'variable' in another discipline, or in another context. To avoid confusion we stick with a single word, 'quantity', that will be used in all disciplines and all contexts. To stipulate differences in meaning we will introduce roles, or categories of quantities in Chapter **??**.



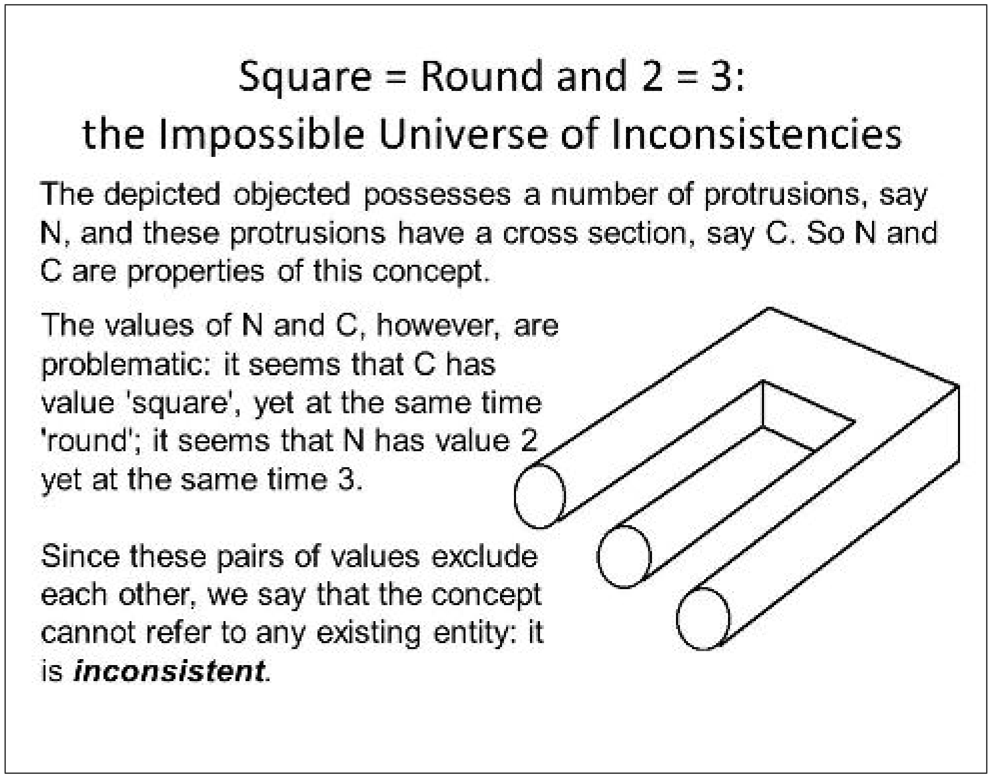
2.6.1 Types of Quantities

Since properties have types, quantities have types, too. We distinguish two sorts of types[10]: ELEMENTARY TYPES and COMPOUND TYPES. To know an element of an elementary type, we don't need any further properties. For instance, integer number 5 is fully known. All properties (such that it is an odd prime, smaller than 19, etc.) of the value 5 can be deduced using nothing else than 'the value is 5'. Other elementary types are BOOLEANS and strings such as 'aAaa', 'aaaa', 'pineapple' or '12345' (the latter not to be confused with the number 12345). We write 'pineapple' for the string consisting of the letters p,i,n,e, ...; `pineapple` without quotes is the name of a concept or a property. Also real numbers, e.g. including π, $\sqrt{2}$, and 12.7,

[10] The image of neon signs was taken from http://commons.wikimedia.org/wiki/Neon_sign#mediaviewer/File:Neon_(12594495033).jpg

form an elementary type. Finally, an elementary type can be an enumerated list of constants. The type `material`, for instance, could be {`wood`,`metal`,`plastic`,`cement`} ▷41 ; the type `shirtNumbersOfFootballPlayers`={1,2,3,4,5,6,7,8,9,10,11}, which is a subset of the set integer numbers, is also a type.

An example of a compound type is the type `rectangle`: the set of all rectangles. To fully know a rectangle, we need further information. Its properties, such as length, width, area, perimeter, orientation, ... should be CONSISTENT[11] , that means: a concept with properties that have the indicated values should be LOGICALLY POSSIBLE. For instance, a rectangle with area 12 and perimeter 6 cannot exists: these two properties have inconsistent values. Further, for the sake of efficiency, properties should be INDEPENDENT, that is: the value of a property should not be derivable from the value of another property. So properties for defining a rectangle could be `width` and `height` both of type `real`. Every non-negative width and non-negative height determine a rectangle. Alternatively, we could give the properties `area` and `perimeter`: indeed, the width and height of a rectangle uniquely follow from the area and perimeter ▷42 .



We saw a notation for concepts in terms of their properties. This works for properties, both of elementary type and compound type.

We give the example of a rectangular box, that is a configuration of six rectangles. For brevity, we omit the units; all lengths are in cm. First, we could write this as Expression 2.1. This expression, although formally acceptable, gives no insight at all. It is difficult to see if the right dimensions are provided for the right edges.

A better way to write this, is Expression 2.2, together with the definition from Expression 2.3. We re-use the values `r1`, `r2`, `r3` so that we can easily see that top and bottom are CONGRUENT - that is, one is the result of translating, scaling or rotating the other. Similar for left and right, and for front and back.

This form still requires verification to see if the dimensions for matching edges are consistent.

[11]The image of an impossible object is taken from `http://commons.wikimedia.org/wiki/File:Impossible.png?uselang=nl`

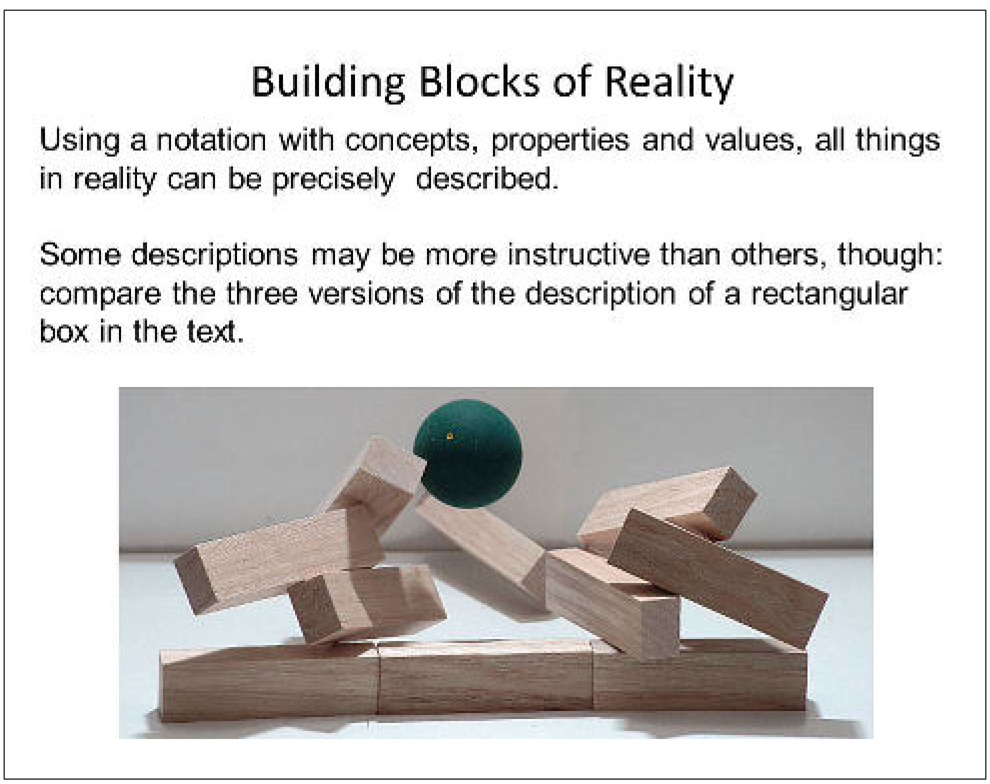
A third attempt is Expression 2.4, which is truly simpler than the earlier two versions.

$$\begin{aligned}\texttt{rectBox} = [&\texttt{bottom:[width:3, height:4], top:[width:3, height:4],}\\ &\texttt{left:[width:4, height:2], right:[width:4, height:2],}\\ &\texttt{front:[width:3, height:2], back:[width:3, height:2]}].\end{aligned} \tag{2.1}$$

$$\texttt{rectBox} = [\texttt{bottom:r1, top:r1, left:r2, right:r2, front:r3, back:r3}]. \tag{2.2}$$

$$\begin{aligned}&\texttt{r1=[width:3, height:4],}\\ &\texttt{r2=[width:4, height:2],}\\ &\texttt{r3=[width:3, height:2].}\end{aligned} \tag{2.3}$$

$$\texttt{rectBox=[width:3, height:2, depth:4]}, \tag{2.4}$$



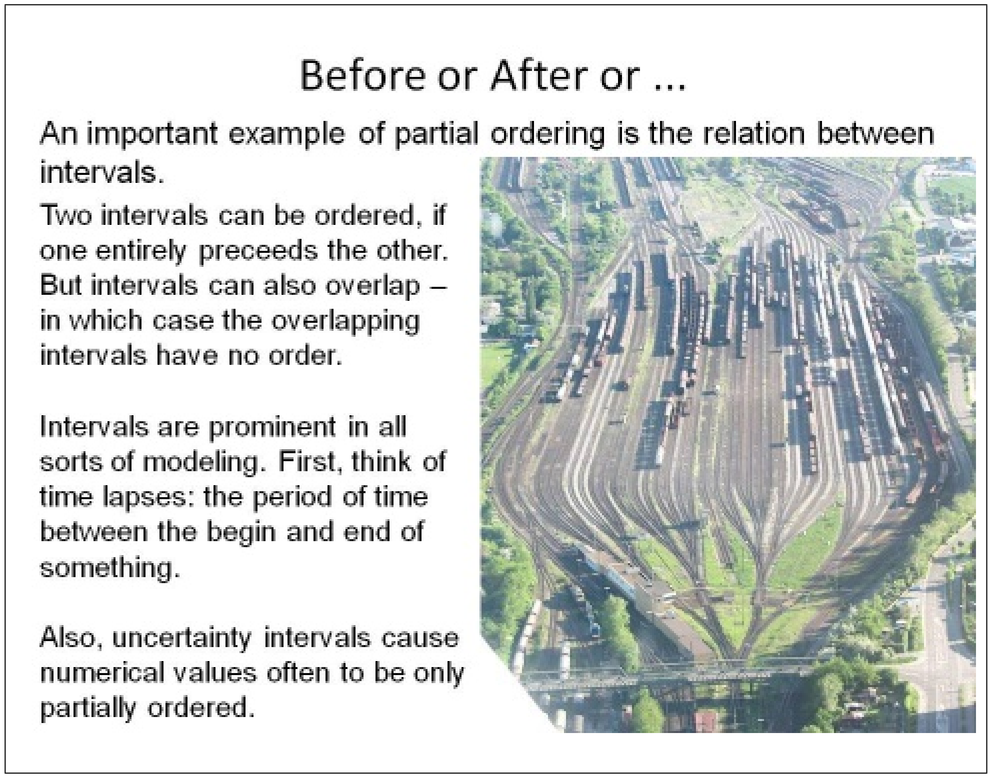
This example[12] shows that we should consciously choose properties and concepts such that the conceptual model represents the entity we want to model. In this case: if we define the box to consist of six rectangles, many CONSTRAINTS , i.e., limitations to the values of various properties, have to be fulfilled. Adjacent sides should have the same length for a shared edge, and a box has 12 such edges. The second version uses the symmetry of the rectangular box. The top and bottom rectangle are the same concept (rectangle r1), so we have to verify fewer constraints. Similar for the two other pairs of faces. The third version is the simplest of all. There are no constraints left, since the length, width and height of the box can be set independently.

[12]The image of blocks was taken from http://commons.wikimedia.org/wiki/Category:Wooden_blocks#mediaviewer/File:Wood_Block_Break_Out.JPG

2.6.2 Operations on Quantities: Ordering

Conceptual modeling preceeds quantitative modeling. In quantitative modeling, we do mathematical operations on quantities. Which operations are allowed for which quantities, however, depends on the types of these quantities.

An important distinction between various types is, whether quantities can be ORDERED. Quantities that can be ordered, are called ORDINAL. Quantities that cannot be ordered are called NOMINAL.



A familiar ordering is the ordering of numbers: $3 < 7$. This relation is TRANSITIVE, that is, from $a < b$ and $b < c$ we have that $a < c$.

Further, it has an opposite relation: the opposite of $<$ is $>$.

Finally, it is defined for *any* two different numbers: always one or the other is the bigger one. This last condition is obvious for numbers, but in most other cases it does not hold[13]. For instance, we can look at family relations between people. My father is an ancestor of me, and his father was an ancestor of him and thereby also an ancestor of me: `ancestorOf` is transitive. The opposite is `descendantOf`. But most people are neither an ancestor nor a descendant of me. Therefore `ancestorOf` only applies to *some* pairs of individuals. This is called PARTIAL ORDERING.

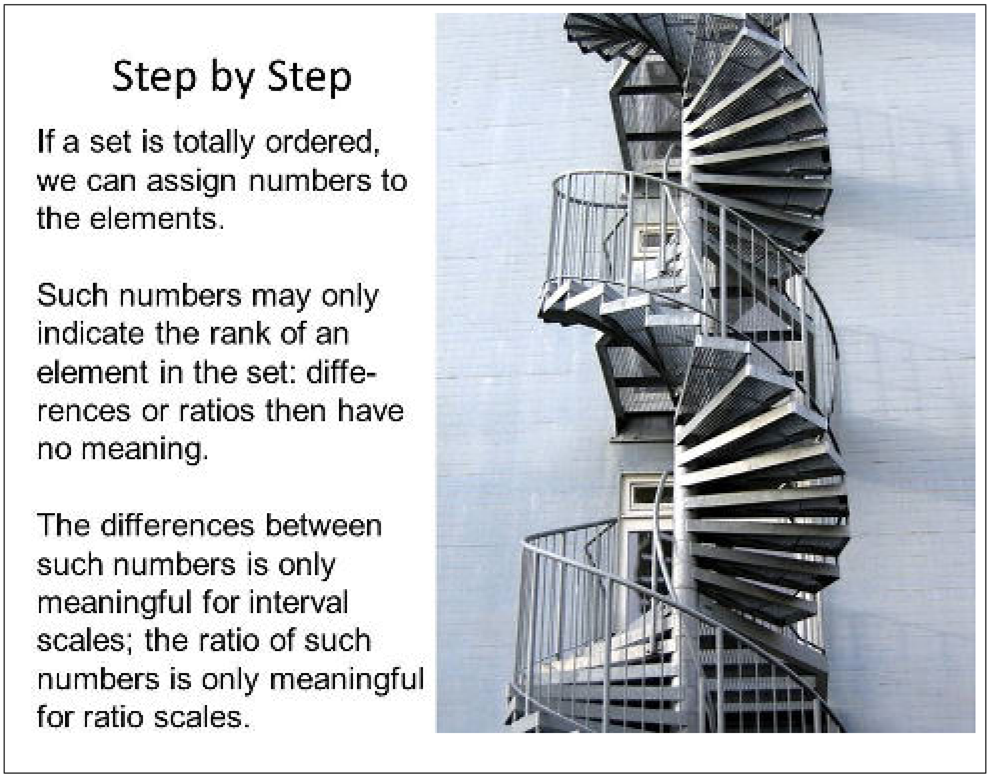
Another, perhaps more important example of concepts that are partially ordered, is *intervals*. Indeed, an interval with an upper bound that is entirely below the lower bound of a second interval is `below` this second interval, but if the two intervals overlap this is not the case. Intervals will play an important role in Chapter **??** when we investigate sensitivity of models.

Sets that are totally ordered are called *totally ordered sets*; sets that are partially ordered are called partially ordered sets or POSETS. Sets that are not ordered at all are called NOMINAL. A set of countries is an example of a nominal set. If we consider a country as a concept with properties such as area, population or gross national product, we can define an ordering, but what we order

[13] The image of a classification yard (Dutch: 'rangeerterrein') showing many overlapping and non-overlapping intervals (sequences of freight cars) was taken from http://commons.wikimedia.org/wiki/File:Rbfkornwestheim.jpg

then is really a set of areas, a set of populations sizes or a set of amounts of money, and not a set of countries.

The various operations that are allowed for each type of set are explained below[14]:



- *nominal sets:* In a nominal set, elements have no ordering. We can only assess if two elements are equal or not, and we can count how many elements (concepts) occur for which some property has some given value. We can say 'in this collection of cars, Volkswagens occur twice as much as Opels', but that does not mean that there is any ordering between a particular Volkswagen and a particular Opel.
- *partially ordered sets:* Transitive relations often give rise to partially ordered sets. Examples are `descendsFrom`, or `comesBefore`. The latter relation occurs for instance if we deal with processes where things happen at different times. See Section 3.3.1 for an example. Partial ordering may allow the verification of a design decision: 'alternative A is better than alternative B'. It occurs when dealing with preferences: you may like chess more than rugby, and chess also more than waterpolo, but the preference between rugby and waterpolo may be unknown.
- *totally ordered sets:* In a totally ordered set an ordering relation exists between *any* two elements in the set. An example is MOHS SCALE for mineral hardness ▷43 . As follows: take two samples of two different minerals; push one firmly onto the other and move. Only one of the two will receive a scratch. This introduces an ordering between any two minerals: `isSofter`, which is the opposite of `isHarder`.

  There are various sorts of scales associated to totally ordered sets ▷44 .

  1. *ordinal scale:* Mohs scale is an example of a totally ordered scale . It is possible, for a set of different minerals, to assign an integer number to each of them. This is also called a RANKING. But it is meaningless to ask if diamond is the same amount harder than copper oxide, as copper oxide is harder than chalk. So, taking averages of Mohs numbers to talk about 'the average hardness' is not allowed ▷45 . For a collection of minerals, however, it is allowed to search for the MEDIAN: the mineral for which the number of minerals that are less hard equals the number of minerals that are harder.

[14] The image of the spiral staircase is taken from http://www.rgbstock.nl/photo/mfjMtZA/wenteltrap+1

OK to compute ...	Nominal	Ordinal	Interval	Ratio
frequency distribution	yes	yes	yes	yes
median	no	total order:yes; partial order: no	yes	yes
add, subtract, mean	no	no	yes	yes
ratio	no	no	no	yes

Table 2.1: Operations allowed on various types of scales

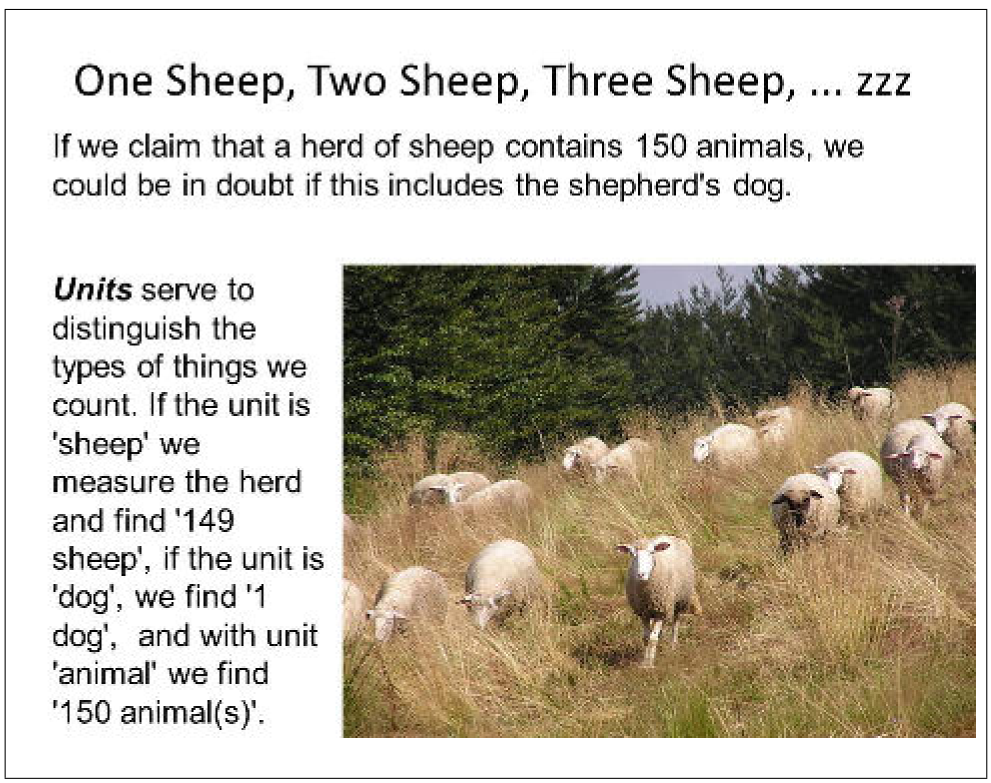
2. *interval scale:* The *difference* between two Mohs numbers has no meaning. For temperatures in a Celsius scale, however, the difference between 10 and 20 degrees Centigrade has a meaning: it corresponds to an amount of energy, and the same amount of energy is needed to heat up something from 80 to 90 Centigrades. Scales that allow addition or subtraction, are called INTERVAL SCALES.

3. *ratio scale:* For the Celsius scale the ratio between, say 80 and 20 Centigrade does not correspond to something physical. For the Kelvin scale, though, a ratio between two temperatures corresponds to a ratio between energy contents: an amount of gas at 80 Kelvin contains 4 times as much energy as the same amount of gas at 20 Kelvin. From this, it follows that the energy contents of *any* amount of gas at 0 Kelvin is 0 Joule. The Kelvin scale is an example of a RATIO SCALE. A ratio scale has a meaningful zero, whereas the zero for a difference scale is arbitrary.

We summarize this in Table 2.1 ▷46 .

2.7 Units, Scales and Dimensions

2.7.1 Counting is Easier than Measuring

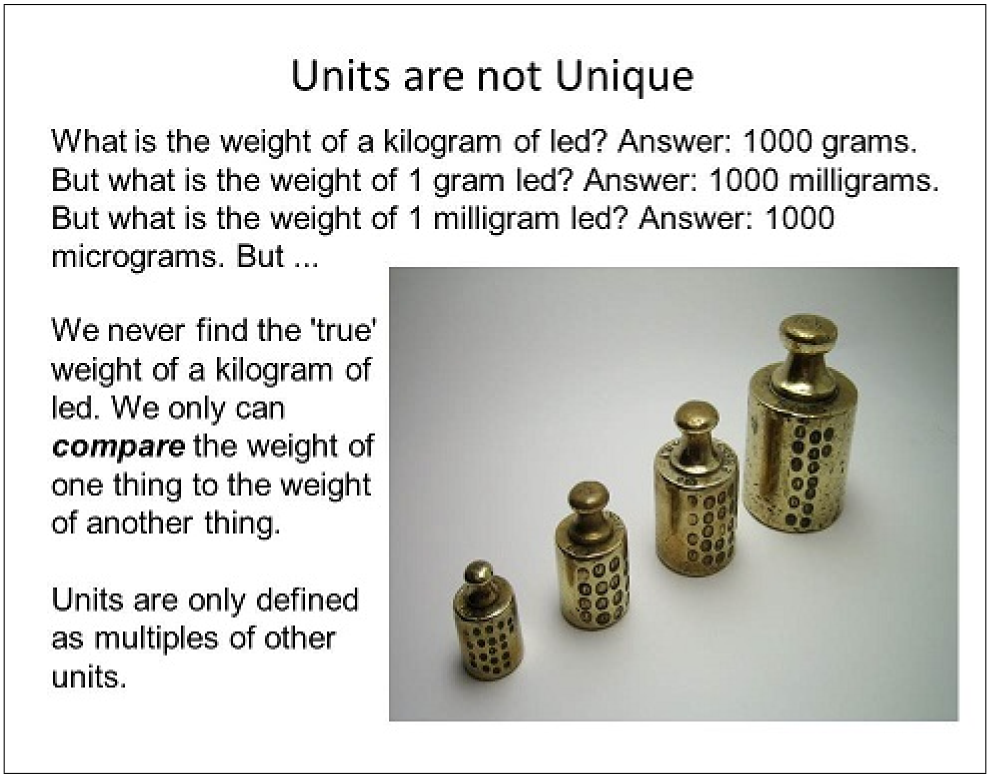


Quantities in a model often correspond to observations or measurements. The simplest form of quantitative observation is counting: answering the question 'how many units of sort X do I have?[15]'.

Suppose that we want to have the dimensions of a piece of land. Assume a known ASPECT RATIO for the piece of land, e.g. square. The piece of land is surrounded by barbed wire, spanned by poles 10 meter

[15] The image of sheep was taken from http://www.rgbstock.nl/photo/mhilDzK/Schapen+in+de+bergen

apart. We can count the poles. If we find 40 poles, we conclude that the perimeter of the piece of land is 400 meter, and the area amounts to 1 hectare.



Here, measuring is reduced to counting[16]. But we could doubt the precision. Are the poles really exactly 10 meter apart? To get more precise results, we use a measuring rod, say of 1 meter length. Again, measuring amounts to counting. The number of times the measuring rod fits in the perimeter is, say, 398 times plus a bit.

To make our result even more precise, we use a shorter unit, of one decimeter long. This time we find 3986 units plus a bit. Repeating the experiment with an even shorter unit (a centimeter) produces 39863 units plus a bit, and so on, until the unit is too small to assess if there still is a remaining bit, or until our curiosity is satisfied, we have run out of time, or we have no smaller units at our disposal.

From this experiment, we learn the following. Suppose that we have two units, u_1 and u_2. They have a ratio $p_{1;2} = \frac{u_1}{u_2}$. That means: unit u_2 fits $p_{1;2}$ times in unit u_1. $p_{1;2}$ is an integer, it counts the number of times u_2 fits in u_1. Next, there is a quantity l, measured with u_1. This gives the number x_1; if we measure the same quantity with u_2 it gives x_2.

So we have $x_1 u_1 = x_2 u_2$, or $\frac{x_2}{x_1} = \frac{u_1}{u_2} = p_{1;2}$. We can use $p_{1;2}$ to predict x_2 if we have measured x_1 (namely: $x_2 = p_{1;2} x_1$), or the other way round.

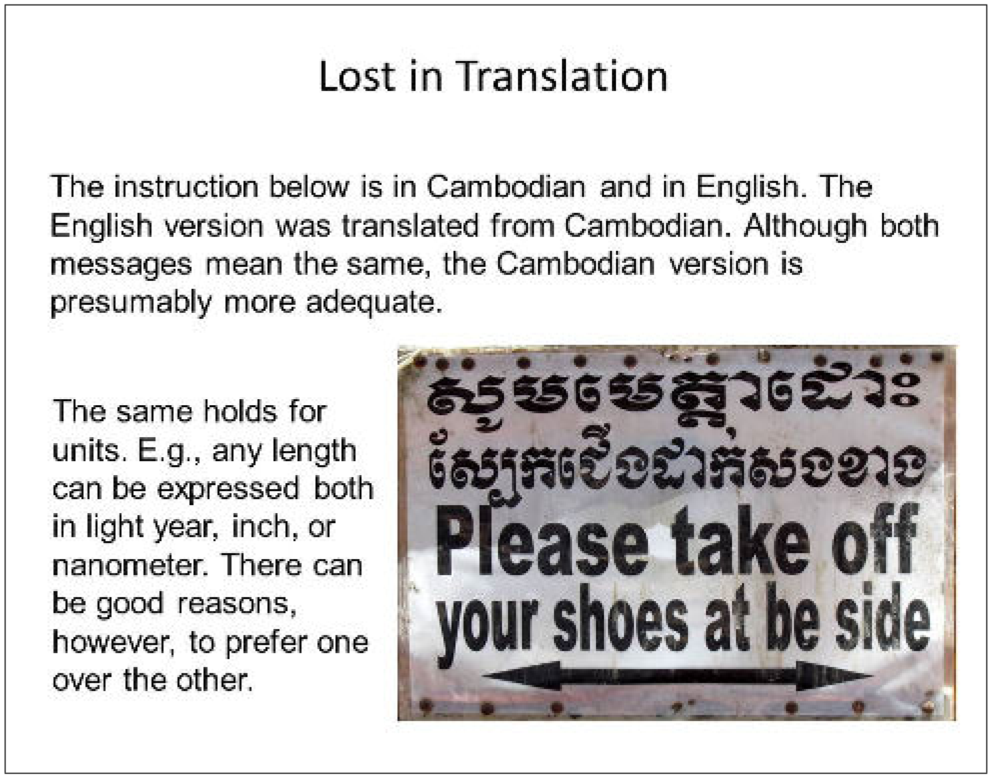
Now let there be a third unit u_3. Then we write

$$\begin{aligned} p_{1;2} &= \frac{u_1}{u_2} \\ &= \frac{u_1}{u_2}\frac{u_3}{u_3} \\ &= \frac{u_1}{u_3}\frac{u_3}{u_2} \\ &= p_{1;3}\ p_{3;2}. \end{aligned} \tag{2.5}$$

[16] The image of the weights is taken from http://www.rgbstock.nl/photo/mWjRlAm/Oma%27s+Old+Weights

So: in order to go from one unit to another unit[17], we use Expression 2.5.

For units u_i and u_j, the number $x_i\ p_{i;j} = x_i/p_{j;i}$ is an INVARIANT of the measured thing. That is: it does not change, if we measure the *same* thing with a *different* measuring unit.



We write that a length is 399 m instead of 399. The expression '399 m' is not just shorthand for '399 measured in meters'. Rather, it is a mathematical expression that involves a multiplication, in the same way that we write ab if we mean $a \times b$.

The factor 'm' in '399 m' is a multiplication with a number $p_{m;U}$ to obtain the invariant $x\ p_{m;U}$, where the number $p_{m;U}$ is different for any unit U. The symbol 'm' means $\times p_{m;U}$ or $p_{m;U}$ for short, where the numerical value of $p_{m;U}$ is not stated.

The *ratio* of the $p_{m;U}$ between two units, 'meter' being one of them, is defined when we know the other unit, U, we want to use. So we don't know what '100 m' means in absolute sense, but we know what it means in comparison with another length expressed in meters, or in comparison with another length expressed in other units. Notice that this ratio does not depend on anything we are measuring: it is purely a property of a pair of units.

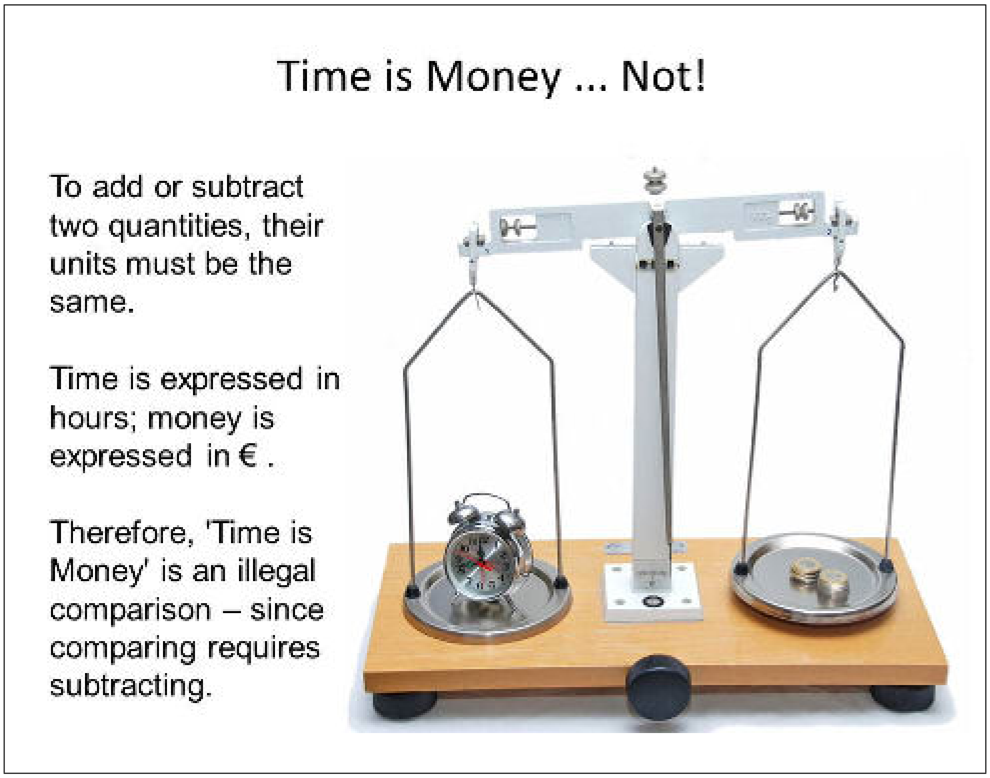
To work with quantities, we don't need to know the numerical value of $p_{m;U}$, or, in general, $p_{i;j}$. If we want to go to another unit, we use Expression 2.5:

$$\begin{aligned} x_m\ m &= x_m\ p_{m;U} \\ &= x_m\ p_{\ m;cm}\ p_{cm;U} \\ &= x_m\ \frac{u_m}{u_{cm}}\ p_{cm;U} \\ &= x_m\ 100\ p_{cm;U} \\ &= 100\ x_m\ cm. \end{aligned} \tag{2.6}$$

So, again, '1 m = 100 cm' states the equality of two algebraic products; the first one is the

[17] The image of the Cambodian-English instruction was taken from http://www.rgbstock.nl/photo/nKv2kXq/lost+in+translation

product of '1' and 'm', and the second one '100' and 'cm'.



Suppose[18] that we want to calculate the area of the piece of land. We measured one side in meters, giving 100 m (being shorthand for $100 \times m$) and for some reason, the other side in decimeters, giving 1000 dm. We know that the area of a rectangle is found by width $\times$ height. In this case we have $(100 \times p_{m;U}) \times (1000 \times p_{dm;U}) = 100.000 \times p_{m;U} p_{dm;U}$ or 100 m $\times$ 1000 dm=100.000 m dm. Although this is consistent, we don't commonly write factors like 'm dm'. We use Expression 2.5 to write $p_{dm;U} = p_{dm;m} \; p_{m;U} = \frac{u_{dm}}{u_m} \; p_{m;U} = 0.1 \; p_{m;U}$ to express this instead as 10.000 m m. This is again an algebraic product, and we write m m=m$\times$m =m^2. So the term 'm^2' is the consequence of consistent algebraic manipulation where symbols such as 'm' are treated as factors, representing unknown factors p with constant ratios.

We summarize:

Unit symbols are algebraic factors that are part of the expression, and should be manipulated as such when doing algebra with the quantities.

Algebraic operations include multiplications, but also additions or subtractions. To express that the perimeter of some piece of land equals twice the length (say, l_l=100 m) and twice the width (say, l_w=70 m), we write:

$$\begin{aligned} \text{perimeter} &= l_l \times p_{m;U} + l_w \times p_{m;U} + l_l \times p_{m;U} + l_w \times p_{m;U} \\ &= 2 \times (l_l + l_w) \times p_{m;U} \\ &= 2 \times 170 \times p_{m;U} \\ &= 340 \text{ m}, \end{aligned} \tag{2.7}$$

where we have explicitly taken the quantity $p_{m;U}$ 'out of the brackets'.

Taking a quantity out of the brackets is only allowed if the same quantity occurs in all the terms between the brackets. Suppose we want to express that we traveled 10 km in a taxi, after waiting half an hour. We might want to write the total waiting time as an addition: 10 km plus 30

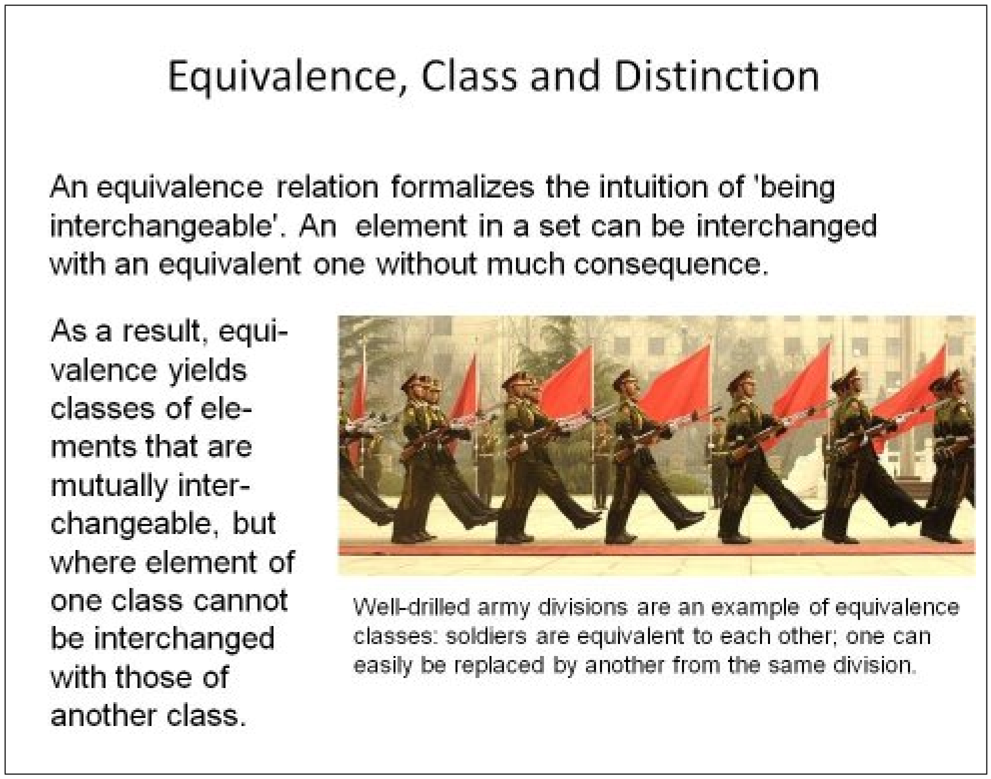
[18] The "time is money"-image was taken from http://www.rgbstock.nl/photo/mhYAppa/Clock+and+money+on+the+weighin

minutes. This is not wrong, but when we try to formalize it in the same way as calculating the perimeter of a piece of land we find:

$$\begin{aligned} \text{time elapsed} &= 10 \times p_{km;U} + 30 \times p_{minute;U} \\ &= \cdots , \end{aligned} \tag{2.8}$$

which cannot be simplified further. There is no common factor that we can take outside the brackets. There is no ratio between $p_{km;U}$ and $p_{minute;U}$ that is independent from any measurement, so we cannot express one as multiple of the other.

Therefore, adding two quantities with different units, say u_1 and u_2 is not forbidden (when we calculated the perimeter of the piece of land we added meters and decimeters), but if the result should be expressed as a quantity with a single unit, say u, u_1 and u_2 must have a constant ratio.



2.7.2 Units and Dimensions

We have seen[19] an example of several units (m, dm, cm) that correspond to factors $p_{m;U}$, $p_{dm;U}$, $p_{cm;U}$, respectively; these units have constant ▷47 ratio's that don't depend on anything measured: $p_{m;U} = 0.1 p_{dm;U}$, $p_{dm;U} = 0.1 p_{cm;U}$, and similar for units km, mm, μm, nm, etc.

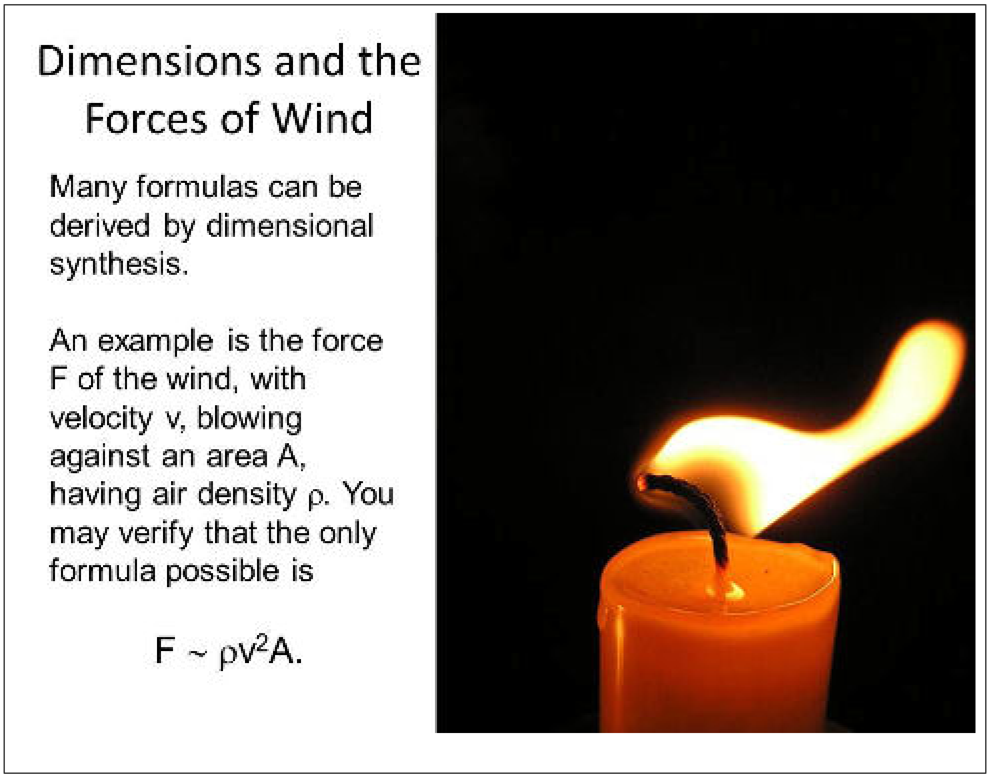
Two units, with p's that have constant ratio's, are called EQUIVALENT ▷48 .

Things that are equivalent can be grouped in so called equivalence classes. Indeed: two things that are not equivalent cannot be in the same class. The equivalence classes, belonging to the relation 'has a constant ratio with' between two units, are called DIMENSIONS. To signify that 'length' is a dimension we usually use an abbreviation and a conspicuous font, like $\mathcal{L}$ for length. Examples of dimensions are length ($\mathcal{L}$), time ($\mathcal{T}$), mass, ($\mathcal{M}$) and many others.

A value is denoted by a number and a unit; the name of a quantity is sometimes annotated by the dimension in square brackets. So, if P.l is the property l of concept P, and it has dimension length, we may write P.l $[\mathcal{L}]$.

[19] The image of a Chinese army parade is taken from http://commons.wikimedia.org/wiki/File:Chinese_honor_guard_in_column_070322-F-0193C-014.JPEG?uselang=nl

Some dimensions correspond only to a single unit, for instance $\mathcal{SHEEP}$ (used to measure the size of a flock of sheep by counting; there are no fractional sheep, so units such as $\mu\mathcal{SHEEP}$ don't occur; we could talk about $k\mathcal{SHEEP}$ to refer to 1000 sheep, though), and another one is $\mathcal{PIANO}$ (used to measure the size of a collection of piano's by counting them).



Some dimensions can be constructed from other dimensions. We have seen the example where the unit of area was expressed as the product m m or m^2. Other units for area are cm^2 or (light year)^2. These are also equivalent: indeed, m^2=10.000 cm^2, etc., so there is a constant ratio. Therefore we associate a dimension to these units, by the name of 'area' or $\mathcal{L}^2$.

The dimension area results by multiplying two equal dimensions ($\mathcal{L} \times \mathcal{L}$). Dimensions can also be constructed by multiplying or dividing *unequal* dimensions. For instance, the dimension 'speed' is $\mathcal{L}/\mathcal{T}$; its units could be km/h, light year/s or μm/month.

We have seen that units can be multiplied and divided, and therefore dimensions can also be multiplied and divided. Units can also sometimes be added, for instance 3m+5dm+7cm. The units m, dm and cm have constant ratio's, and therefore they are equivalent. So they correspond to the same dimension, $\mathcal{L}$. This dimension can again be taken outside the brackets: if we have a quantity q=3m+5dm+7cm, then the dimension of q is $\mathcal{L}$ $^{\triangleright 49}$.

2.7.3 Dimensions and Formulas

If two quantities are equal, their dimensions are also equal $^{\triangleright 50}$. This means that we can, to a large extent, guess the form of mathematical expressions, merely by observing dimensions.

We illustrate this for a pendulum[20].

A pendulum is a weight on a chord, subject to gravity. The oscillation time T of a pendulum is an amount of time T [$\mathcal{T}$]. It could depend on the mass of the weight, m [$\mathcal{M}$], the length of the chord, l [$\mathcal{L}$], and the gravity acceleration, g [$\mathcal{L}\mathcal{T}^{-2}$]. Suppose that the expression we are looking

[20] Another example is illustrated in the preceding image box. The candle flame image is taken from http://www.rgbstock.nl/photo/mC2yjPI/kaarsen+3

for reads

$$T = m^{\alpha} l^{\beta} g^{\gamma}, \tag{2.9}$$

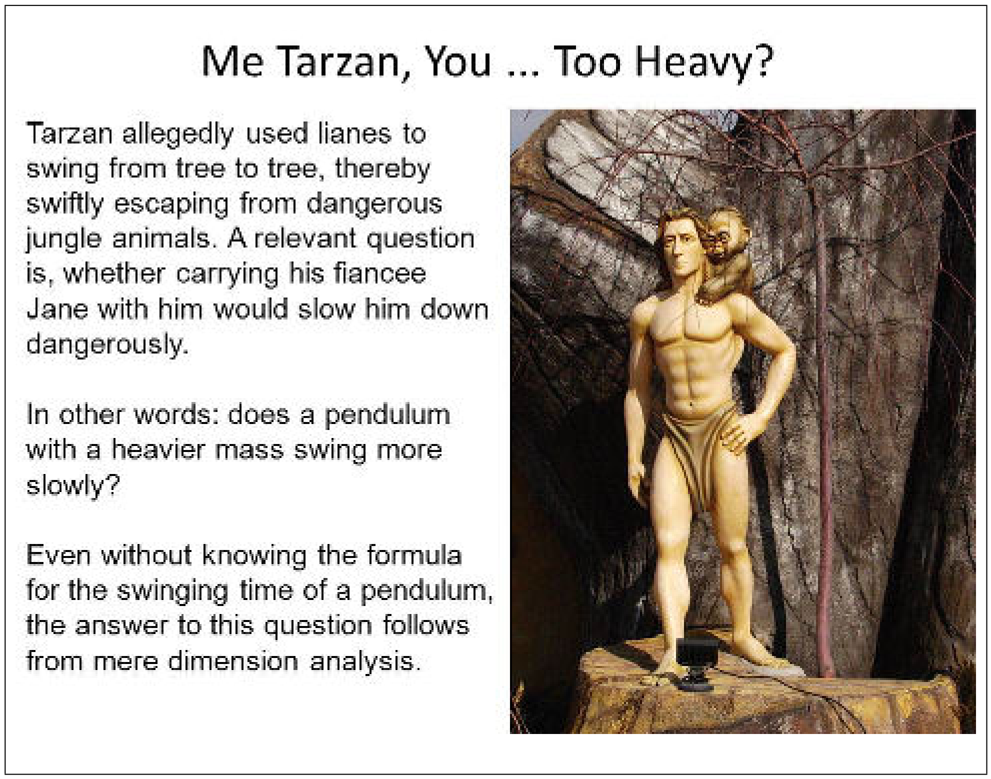
then we must find α, β, and γ. We equate the dimensions left and right; moreover, we use the fact that different dimensions are no multiples of each other ▷51 . Substituting the dimensions for mass, length and acceleration, we get[21]

$$\begin{aligned} \mathcal{T} &= \mathcal{M}^{\alpha}\mathcal{L}^{\beta}(\mathcal{L}\mathcal{T}^{-2})^{\gamma} \\ &= \mathcal{M}^{\alpha}\mathcal{L}^{(\beta+\gamma)}\mathcal{T}^{-2\gamma}. \end{aligned}$$

Equating the powers for $\mathcal{T}$, $\mathcal{M}$ and $\mathcal{L}$ we get:

$$\begin{aligned} \mathcal{T}: \quad 1 &= -2\gamma; \\ \mathcal{M}: \quad 0 &= \alpha; \\ \mathcal{L}: \quad 0 &= \beta + \gamma. \end{aligned}$$

So $\alpha = 0$, $\gamma = -\frac{1}{2}$, and $\beta = \frac{1}{2}$. The expression for the oscillation time therefore must have the form $T \propto \sqrt{\frac{l}{g}}$, consistent with the high-school formula $T = 2\pi\sqrt{\frac{l}{g}}$.



2.8 Mathematical Tools: Functions of Two Variables Continued

In this chapter we discuss the conceptual model, consisting of concepts, properties and the relations between them. Relations often state that one quantity depends on several others. In Section 1.7 we introduced functions of multiple variables to help formalize such dependencies. In the same chapter, we saw that a common modeling purpose is *optimization*. We should be able, therefore, to optimize functions of several variables. In this section we will see that optimizing often entails to finding extreme values of functions, and we give some mathematical background for finding such extreme values.

[21] Tarzan's portrait was taken from http://commons.wikimedia.org/wiki/Category:Tarzan#mediaviewer/File:Harikalar_Diyari_Tarzan_06007_nevit.jpg

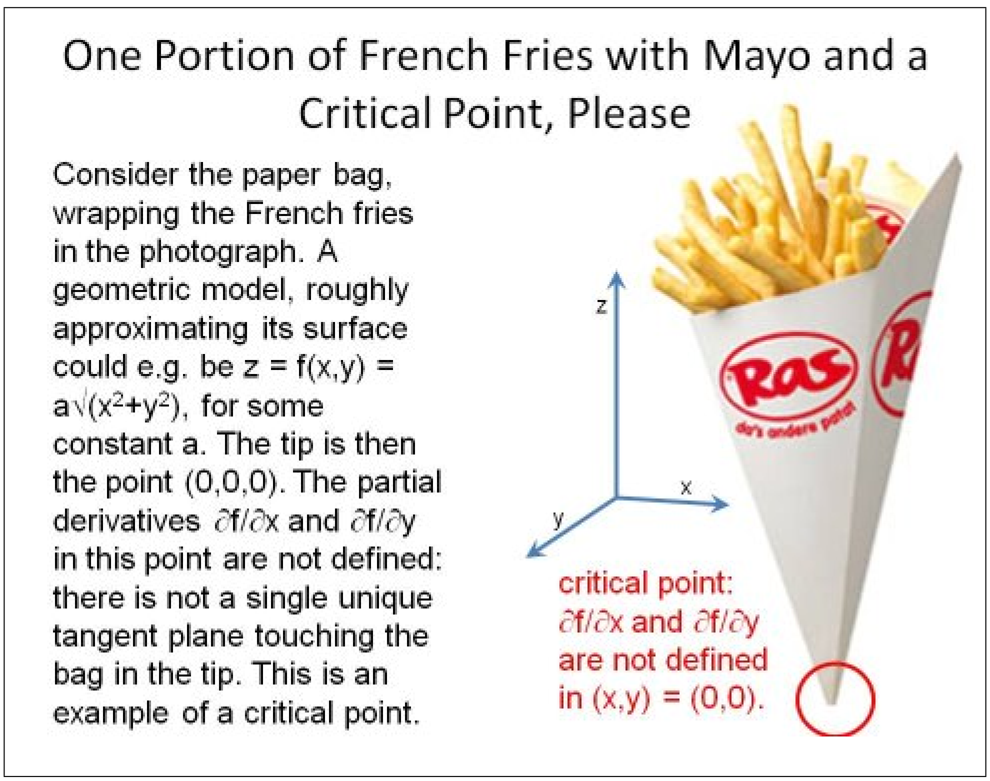
2.8.1 Extrema of Functions of several Variables

Local Extrema and Critical Points

First we recall some definitions and theorems for the case of a function of one variable.

A CONTINUOUS function f defined on a CLOSED, *bounded* interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval [▷52] .

$f(c)$ is called a LOCAL (OR RELATIVE) MAXIMUM of f if $f(c) \geq f(x)$ for all x in some *open* interval containing c.
$f(c)$ is called a LOCAL (OR RELATIVE) MINIMUM of f if $f(c) \leq f(x)$ for all x in some *open* interval containing c.



A point [▷53] c in the domain of f is called a CRITICAL *point* of f if $f'(c) = 0$ or $f'(c)$ is undefined[22].

Suppose that $f(c)$ is a local extremum (maximum or minimum). Then c must be a critical point of f.

Suppose that f is continuous on the *closed* interval $[a, b]$. Then, each absolute extremum of f must occur at an endpoint (a or b) or at a critical point.

Bearing this in mind we can define also extrema for the case of a function of two variables.
The value $f(a, b)$ is called a LOCAL (OR RELATIVE) MAXIMUM of the function f if there is an OPEN DISK C centered at (a, b), for which $f(a, b) \geq f(x, y)$ for all $(x, y) \in C$.
The value $f(a, b)$ is called a LOCAL (OR RELATIVE) MINIMUM of the function f if there is an open disk C centered at (a, b), for which $f(a, b) \leq f(x, y)$ for all $(x, y) \in C$.

It can be seen in the Figure 'Contemplating a Local Extreme' that the tangent plane to a graph $z = f(x, y)$ at the local maximum or a local minimum is a horizontal plane. This means that both partial derivatives there must be equal to zero. However, partial derivatives do not always exist. This reminds us of the case of single-variable functions: to refer to a point with zero derivative or undefined derivative, we used the term 'critical point'. We now generalize critical points to the

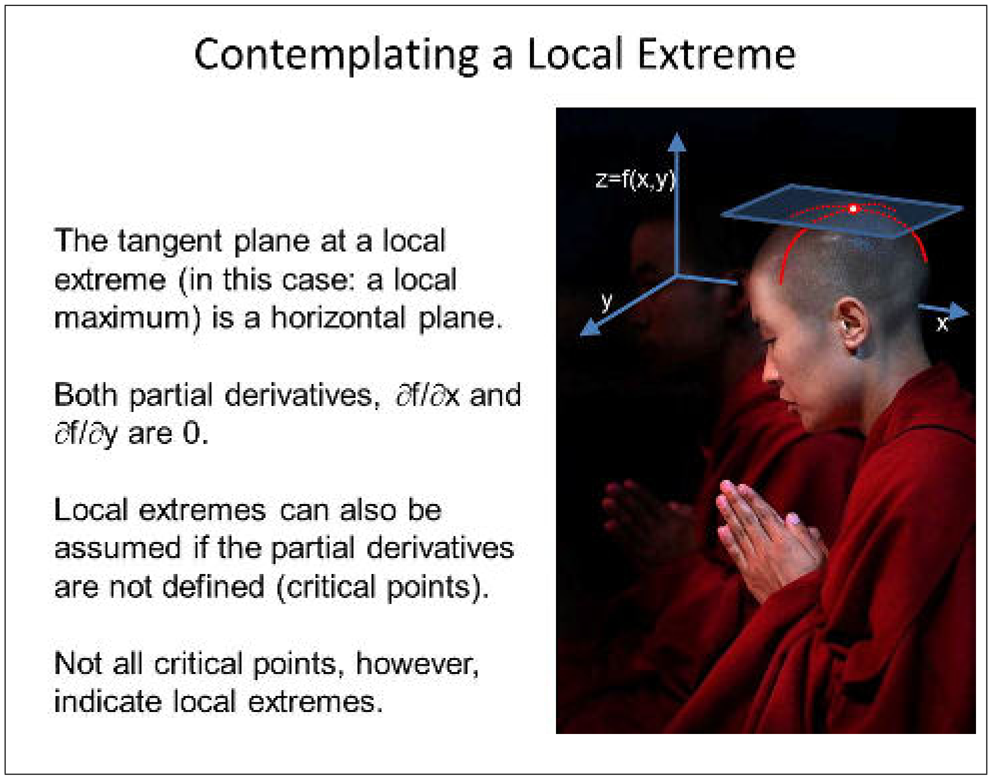
[22]The photograph of a bag of French fries was taken from http://commons.wikimedia.org/wiki/File:RAS_frietzak.jpg?uselang=nl

case of functions with multiple variables:

The point (a, b) in the domain of f is called a *critical point* of the function $f(x, y)$ if either both partial derivatives are zero in the point (a, b) or at least one of the partial derivatives does not exist.

Go to this link to interactive explore a function surface $z = f(x, y)$ and investigate the relations between partial derivatives, tangents, the tangent plane and the normal vector.

Now we can formulate the following *theorem*[23].



If $f(x, y)$ has a local extremum at (a, b), then (a, b) must be a critical point. However, one should realize that critical points are only candidates to give an extremum. The theorem above is not 'if and only if', it is 'if..., then'. Some critical points are not extremal. The Figure 'There is Nothing Extreme in a Pringle' gives some examples. Here we can see a so-called saddle point. For instance, in the pringle-shaped surface $f(x, y) = x^2 - y^2$, both partial derivatives are equal to zero at $(0, 0)$. However, if we take the intersection of the function and the plane $x = 0$, then the function attains a maximum for $y = 0$. If we take the intersection of the function and the plane $y = 0$, then the function attains a minimum for $x = 0$. The same point acts both as a maximum (in dependence of x) and as a minimum (in dependence of y), and therefore is neither of the two according to our definition.

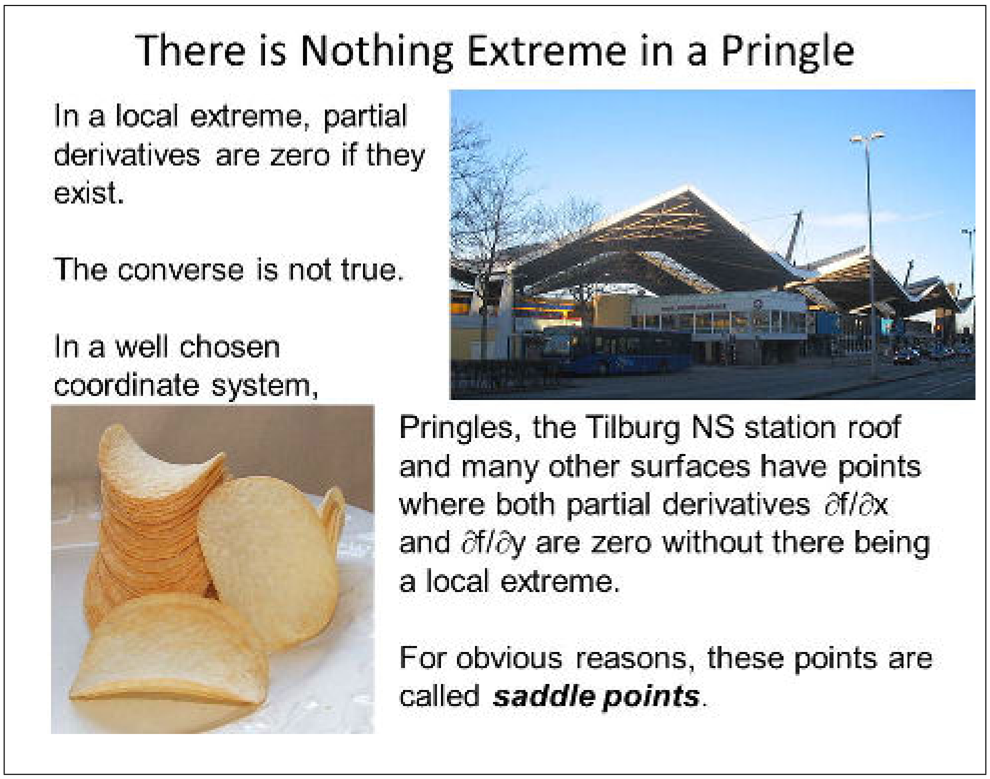
One can use visual inspection to infer whether a critical point is an extremum or not. There is also a mathematical test to determine whether a critical point with both partial derivatives equal to zero gives an extremum or not (the second derivatives test), but we will not discuss it in this section ▷54 . Sometimes also other arguments can be used (see Example 1 below).

To analyse the geometry of a saddle-type surface, click here. The shape of saddle surfaces can also be understood by studying its contour plot. To do so, click here. Presumably the simplest saddle-type surface is $f(x, y) = x^2 - y^2$; the part of this surface where $x^2 + y^2 < c$, for some constant c, could be an adequate model of a pringle. There are many more surfaces that feature saddle points, though. Practical examples occur in furniture such as chairs that have to give a

[23]The photograph of a meditating monk is taken from http://commons.wikimedia.org/wiki/File:Contemplative_Buddhist_monks_from_Bhutan_-_Flickr_-_babasteve.jpg?uselang=nl

comfortable fit to organic shapes found in the (human) body.

A simple example is the surface $f(x,y) = x^3 - pxy$ for various values of p. Click here to investigate the shape of the surface. To get a better understanding of the 3D shape of the surface[24], the orientation can be adjusted by dragging the mouse inside the image, rotating it over x, y or z-axes. The contour plot can be studied by clicking here.



Example 1 The point $(0,0)$ is the one and only critical point of the function $f(x,y) = \sqrt{x^2+y^2}$. Since $f(0,0) = 0$ and $f(x,y) \geq 0$, the value $f(0,0)$ is a minimum.

Example 2 The critical points of the function F in Expression 1.3 are $(-\frac{5}{3}, \frac{5}{3})$, $(-3,-3)$ (for these points both partial derivatives are zero). For all points (R_1, R_2) with $R_1 + R_2 = -3$ the partial derivatives do not exist. However, these are not critical points because they are not in the domain of the function. The value $f(-\frac{5}{3}, \frac{5}{3}) = \frac{112}{77}$ is a local minimum. The value $f(-3,-3) = 0$ is a local maximum. This can be seen here using contour-plots ▷55 . Notice that it may require quite some careful tweaking to find appropriate sets of contour values to actually see the behavior near the critical points: realize that the range of the function from Expression 1.3 is huge: for $R_1 + R_2$ close to -3 it varies from plus to minus infinity, whereas the values in $(-\frac{5}{3}, \frac{5}{3})$ and $(-3,-3)$ are close to 0. So the chance that an arbitrary level curve passes through an 'interesting' region of the domain is quite small. This illustrates a practical limitation to the use of numerically calculated level curves for investigating the behavior of functions of two variables.

Global Extrema

Next we will discuss global (or absolute) extrema.

We call $f(a,b)$ a GLOBAL (OR ABSOLUTE) MAXIMUM of f on the region R if $f(a,b) \geq f(x,y)$ for all $(x,y) \in R$.

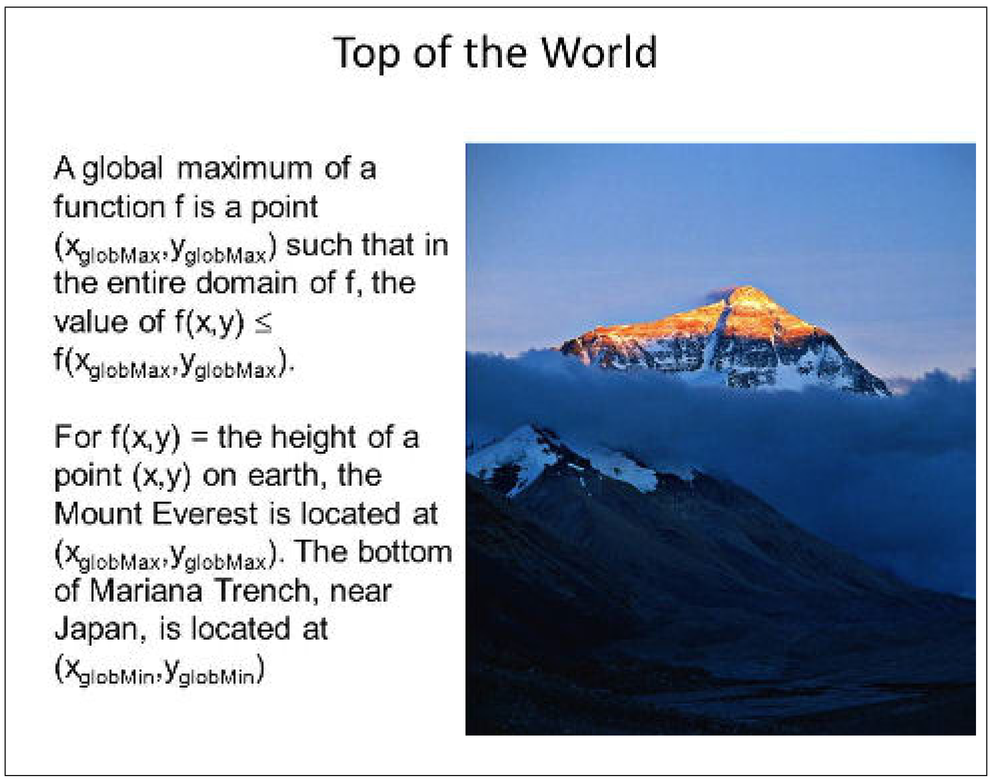
We call $f(a,b)$ a GLOBAL (OR ABSOLUTE) MINIMUM of f on the region R if $f(a,b) \leq f(x,y)$

[24] The photograph of pringles is taken from http://commons.wikimedia.org/wiki/File:Pringles_chips.JPG?uselang=nl; the Tilburg station photograph is taken from http://commons.wikimedia.org/wiki/File:Centraal-Station_Spoorlaan_Tilburg_Nederland.JPG?uselang=nl

for all $(x, y) \in R$.

As for functions of one variable an *extreme value theorem* exists. For these theorem one needs a definition of 'closed' and 'bounded' in two dimensions. Intuitively, a closed region is a region that contains its boundary points. A formal definition of a closed region and of boundary points is more complicated ▷56 . A region R is called bounded if there is a disk that completely contains it. Now we can formulate the following theorem: let $f(x, y)$ be a continuous function, defined on a closed and bounded region. Then f possesses a global maximum and a global minimum, both either in critical points inside R or at its boundary. We will not use this theorem explicitly in the following section, but use graphical methods (contour plots) to find extrema [25].

2.8.2 Constrained Optimization



In Section 1.6 an example is given how the modeling process can be executed in practice. A crucial question that was mentioned in that section was 'with how little money can we safely illuminate the motorway?' The costs of illumination can be given as function of the quantities chosen. But the word 'safely' implies some constraints. The resulting illumination must satisfy the specification for a safe illumination.

There can be two kinds of constraints. The constraint might be an EQUALITY or an INEQUALITY.

An equality constraint is an equation; the found optimum should be such that it solves the equation.

An inequality constraint is an inequality; the found optimum should be such that the inequality is satisfied ▷57 .

For the case of *equality constraints* there exists a mathematical method called the method of Lagrange multipliers. This method is widely applicable ▷58 It is a bit technical, though; here we will consider problems where a slightly simpler - though less general - technique is used. This technique amounts to substituting the constraint equality in the function that is to be optimized.

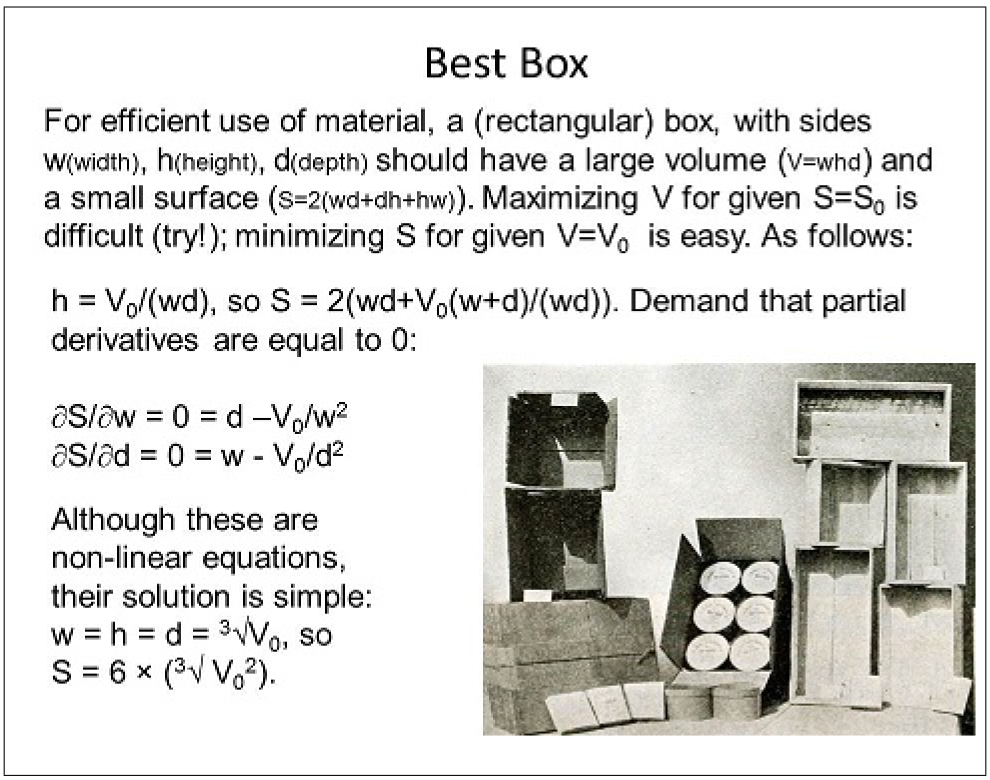
Optimization problems with *inequality constraints* are generally approached by attempting to

[25] The photograph of Mount Everest was taken from http://commons.wikimedia.org/wiki/Mount_Everest#mediaviewer/File:Mount_Everest_(topgold).jpg

transform the inequality constraints to equality constraints.

We proceed with some examples.

Example 1 (continued from Section Local Extrema and Critical Points): equality constraints. Consider again the function $f(x,y)=\sqrt{x^2+y^2}$. We want to minimize this function under the constraint $2x+y=3$. This can be solved us by substituting the equality constraint into the function to be minimized. Then the latter function reduces to a function of merely one variable. We find $g(x)=\sqrt{x^2+(3-2x)^2}=\sqrt{5x^2-12x+9}$. Minimizing $g(x)$ gives $x=\frac{6}{5}$ and so $y=\frac{3}{5}$. The minimum is equal to $\frac{3}{10}\sqrt{15}$; we easily verify that the solution indeed satisfies $2x+y=3$. Another example is illustrated in [26] Figure "Best Box".



Example 2 (continued from Section Local Extrema and Critical Points): inequality constraints. Earlier, we introduced the function F in Expression 1.3, representing the average waiting time in the traffic lights model. Constraints in the traffic light model turn out to be inequalities. We will now introduce a rather general method to find an extremum in the case of inequality constraints. It amounts to reducing the optimization problem to a problem where inequality constraints are re-written in the form of equality constraints.

For the traffic lights model we have the constraint that any traffic light should be green long enough to ensure that the entire queue of waiting cars is resolved at the moment that the traffic light turns red again. In Appendix **??** where Expression 1.2 is derived it is shown that this so-called 'no queue condition' leads to the following inequality constraints (f_0, f_1, f_2 in cars/minute; R_0, R_1, R_2 in minutes):

$$f_0 \geq f_1+f_2, \quad f_0R_2 \geq (2R_0+R_1+R_2)f_1 \quad \text{and} \quad f_0R_1 \geq (2R_0+R_1+R_2)f_2. \tag{2.10}$$

For the values of $f_0=40$, $f_1=\frac{40}{3}$, $f_2=5$, and $R_0=1.5$, used to derive Expression 1.3, this gives

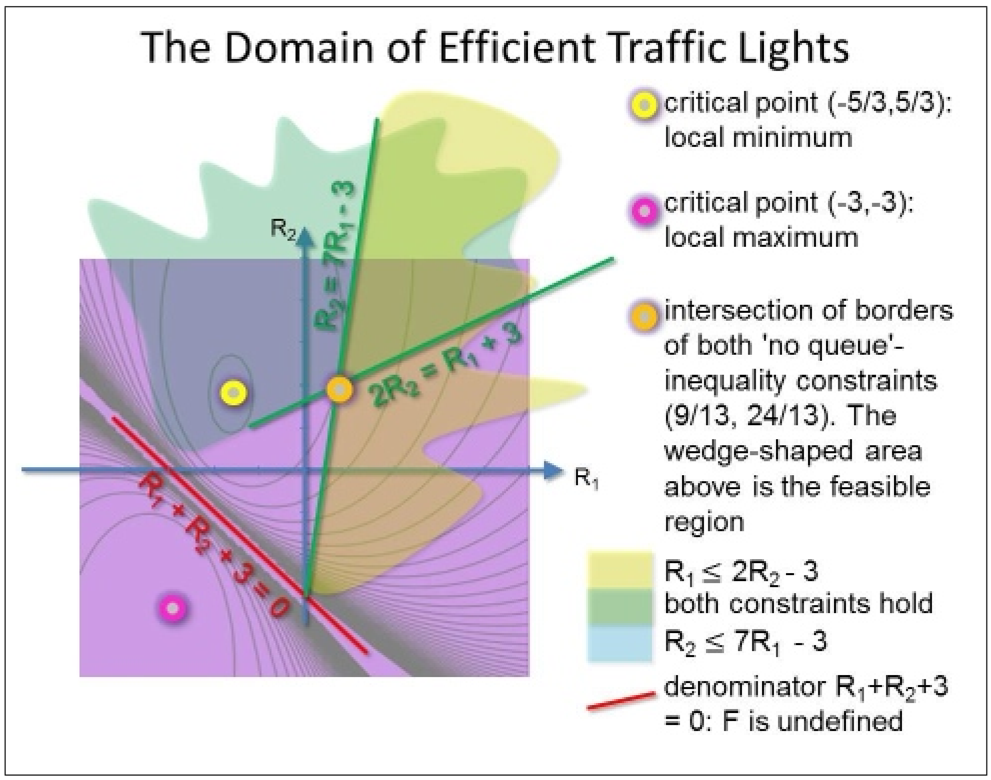
$$2R_2 \geq 3+R_1 \quad \text{and} \quad 7R_1 \geq 3+R_2. \tag{2.11}$$

[26] The photograph of various cardboard boxes is taken from http://commons.wikimedia.org/wiki/File:US_Dep_Agriculture_Bullettin_N_456_Marketing_Creamery_Butter_Fig_10.png

In Figure 'The Domain of Efficient Traffic Lights' both lines, $2R_2 = 3 + R_1$ and $7R_1 = 3 + R_2$ are indicated. These lines are borders of the regions where each of the two inequalities hold (the yellow and blue zones); both inequalities hold in the overlapping wedge (green zone). This is the so-called FEASIBLE REGION. The corner point of the feasible region is the intersection of $2R_2 = 3 + R_1$ and $7R_1 = 3 + R_2$, that is the point $(\frac{9}{13}, \frac{24}{13})$.

We see that none of the two critical points, calculated in Section 2.8.1, i.e. the local maximum of F in $(\frac{-5}{3}, \frac{5}{3})$ and the local minimum of F in $(-3, -3)$, are in the feasible region. There are no critical points at all within the feasible region.

The only place, therefore, where a local minimum of F possible could occur is *on the boundary of the feasible region*.



The feasible region is the region where all *inequality* constraints together hold; its boundaries are lines where *equality* constraints hold. So now we can apply the method that we used in example 1, that is: we substitute each of the equality constraints ($2R_2 = 3 + R_1$ and $7R_1 = 3 + R_2$), into F from expression 1.3, yielding the two 1-variable functions $F_1(R_1) = \frac{6}{77}\frac{105R_1^2+42R_1+63}{8R_1}$ and $F_2(R_2) = \frac{6}{77}\frac{30R_2^2+12R_2+18}{3R_2}$. For each of these two, critical points can simply be found, yielding the solutions $R_1 = \sqrt{\frac{3}{5}}$; $R_2 = 7\sqrt{\frac{3}{5}} - 3$ and $R_2 = \sqrt{\frac{3}{5}}$; $R_1 = 2\sqrt{\frac{3}{5}} - 3$.

Of these two candidate solutions, we see that the second one is entirely outside the feasible region, so the final solution for the traffic light problem is that $R_1 = \sqrt{\frac{3}{5}} \approx 0.77$ minutes and $R_2 = 7\sqrt{\frac{3}{5}} - 3 \approx 2.42$ minutes.

At this point, we may ask if we are certain that there is no way to achieve an even shorter average waiting time. If we inspect the contour plots in the picture 'The Domain of Efficient Traffic Lights' above, it is clear that there it is not possible to achieve a shorter waiting time. This can be proved mathematically by use of the theorem mentioned in Section 2.8.1.

The situation, as in the above traffic lights example, where constraints occur in the form of inequalities, is quite common. For instance, in the street illumination model, the height of the lamps should have a lower bound and an upper bound. The same holds for the illumination on the surface of the road.

Mathematically we have found optimal settings. However, as a modeler we also need to pay attention to the result. The optimal value is attained at a boundary point and one should realize the consequences. At the boundary the 'no queue condition' is met, but a small disturbance might have the result that there will be a queue. And one must realize that it is just a model. We assumed deterministic behavior of the cars. Of course this is not true in reality. Therefore one should validate the model. Here it 'only' concerns waiting times. But think of the consequences if this holds for a model for the design of a nuclear plant.

It is instructive to recapitulate the approach to follow in such cases:

- *find feasible region* : what is the part of the domain where all inequality constraints are satisfied? Check if the feasible region is non-empty - otherwise, there is no solution to the problem. Each part (=segment) of the border of the feasible domain is the solution of one of the constraints, written as *equality* constraints.
- *for each of the segments, substitute the associated equality constraint into the function to be optimized* . The result is an unconstrained optimization problem, and we can search for critical points in the standard way by calculating (partial) derivatives and setting these to 0.
- *for each of the critical points found, check if they are in the feasible region* . If so, these are candidate solutions.
- *check if there are any critical points inside (=not on the borders of) the feasible region* , or prove that there aren't. If there any, add them to the set of candidate solutions.
- *check which of the candidates is the most extreme one* . The latter is the final solution of the constrained optimization problem ▷59 .

Finally, notice that, in general, many constraints can be given: unlike in the case of solving equations, where as a rule the number of equations should equal the number of unknowns to achieve a unique solution, there is no immediate relation between the number of inequality constraints and the number of occurring quantities.

This completes our treatment of analytic, i.e. symbolic, methods for finding local and global extrema, both in the case of equality and inequality constraints, for a function of multiple variables. These methods rely on the function being differentiable. In actual modeling situations, this assumption is often not valid. In later chapters, we will learn methods that also work in the case where the function to be optimized is not differentiable, or in the case where we have several functions that need to be optimized simultaneously.

2.9 Summary

- The *conceptual model* is constructed in stage 2 (conceptualization) of the modeling process;
- The conceptual model consists of *concepts*; *entities* in the modeled system are represented by concepts;
- A concept is a *bundle of properties*, every property consisting of a *name* and a *set of values*: this set is the *type* of the property;
- Concepts can have *relations*; the concepts and relations together form the *conceptual model*, usually drawn as an entity-relation graph. Relations can also exist between the properties

of concepts. The conceptual model is constructed in 4 steps:

- establish concepts;
- establish properties;
- establish types of properties;
- establish relations.

- Sets of values can be *bound* in different ways to properties, e.g. as choices, as results from measurements, or as desired outcomes;
- Values, occurring in the type of a property, can be concepts of their own;
- *Quantities* are properties, where the concept they are properties of is disregarded;
- Allowed mathematical operations on quantities depend on their *ordering*; we distinguish *nominal* (no order), *partial ordering* or *total ordering*. For totally ordered scales, we further distinguish *interval scale* and *ratio scale*;
- *Measuring* amounts to *counting* the number of units of some sort that fit in the measured item. Units can have constant ratio's (e.g., 1m=100cm);
- Sets of units that have a constant ratio are called equivalent. A *dimension* is an equivalence class on units;
- Operations on units follow the operations on quantities (dimensional analysis);
- Using the dimension of quantities, the form of a mathematical relation between them can often be derived (dimensional synthesis).
- Functions of two variables:
  - The value $f(a, b)$ is called a *local (or relative) maximum* of the function f if there is an *open disk* C centered at (a, b), for which $f(a, b) \geq f(x, y)$ for all $(x, y) \in C$. The definition of a minimum is similar.
  - The point (a, b) in the domain of f is called a *critical point* of the function $f(x, y)$ if either both partial derivatives are zero in the point (a, b) or at least one of the partial derivatives does not exist.
  - If $f(x, y)$ has a local extremum at (a, b), then (a, b) must be a critical point.
  - We call $f(a, b)$ a *global (or absolute) maximum* of f on the region R if $f(a, b) \geq f(x, y)$ for all $(x, y) \in R$. Similar for a global minimum.
  - If $f(x, y)$ is continuous on the *closed and bounded region* $R \subset \mathbb{R}^2$, then the function f has both a global maximum and a global minimum on R. Moreover, a global extremum occurs only at a critical point or at a *boundary point* of R.
  - An optimization problem with an equality constraint can very often be solved by substitution of the equality constraint into the function.
  - An optimization problem with inequality constraints can often be reformulated such that it reduces to a optimization problem on a closed and bounded region, the boundaries of which are defined by the inequality constraints.

2.10 Learning goals

2.10.1 Knowledge

You should know the meaning of the terms concept, property, value, type, relation, quantity; the various kinds of ordering and the mathematical operations that are allowed for each form of ordering. You should know and understand the mathematical operations on units, and the various rules that apply. You should understand the concept dimension and dimension analysis / dimension synthesis. You should possess a working knowledge of the optimization of real functions of two variables as explained in Section 2.8 and of the relevant sections in the calculus book of either *Adams* or *Smith & Minton* (see below).

Adams: §13.1 without the second derivative test. §13.2 and §13.3 deal with constrained optimization. However the method of Lagrange multipliers is applied and this not part of the material for the exam. You must understand the basic ideas given in Section 2.8.2.

Smith & Minton: §12.7 without the second derivative test and without the method op steepest ascent. §12.8 deals with constrained optimization. However the method of Lagrange multipliers is applied and this not part of the material for the exam. You must understand the basic ideas given in Section 2.8.2.

2.10.2 Skills

In this section, with 'problem' we mean: a problem that does not require domain-specific knowledge exceeding your present knowledge.

For a conceptual model, needed to solve a problem in some domain, you should be able to identify the most important concepts and their properties, and you should be able to denote the values for these properties using set notation. You should be able to construct an entity-relation graph depicting the concepts and the most important relations between them. You should be able to assess the types of the occurring quantities, determine if they can be ordered and in what sense, and you should be able to make a justified choice for the units to use. You should be able to convert arbitrary formulas from one unit system to another.

You should be able to check derivations and formulas using dimensional analysis, and in simple cases you should be able to derive formulas using dimension synthesis.

For a given function of two variables, you should be able to find the critical point of the function for a given region, classify the critical points and find the extrema. You should also be able to find the extrema when (in)equality constraints are present, applying the basic ideas given in Section 2.8.2.

2.10.3 Attitude

When confronted with a problem that might benefit from a formal approach, you should consider to use a model. When approaching a problem by using a model, you should have the attitude to build a conceptual model. You should have the inclination to formulate the properties of the occurring concepts in terms of quantities with well-defined types, and you should typically denote these in terms of set-notation. Whenever you encounter a formula, you should check its dimensional consistency. If you need to optimize a function, you should consider the possibility of using the analytical tools given in this chapter.

2.11 Questions

1. We use the terms 'entity' and 'concept'. Explain in your own terms what the difference is between them.
2. (*) Is it possible to talk about an entity that is not a concept?
3. What can you say about the arity of `isA`, `hasA`, `specializesTo` and `partOf`?
4. Is it possible to access a concept that has no name?
5. Consider the words 'color', 'tomato', 'red'. Which is a concept, which is a property name, which is a value, which is a type, which is a function?
6. If 'red' is a concept, give some examples of properties and values. Can you think of a property of 'red' such that 'tomato' is a value?
7. We state: 'A property is a function of the concept it belongs to'. Explain.
8. Explain in your own words the meaning of 'intersubjective'.
9. Explain in your own words the meaning of 'segmentation'.
10. What is the difference between $\{4\}$ and 4 and '4'?
11. What do we mean by 'opposite relations'?
12. What is transitivity?
13. What is the meaning of a 'range', when we use this term in defining a type?
14. What are the 4 steps of constructing a conceptual model?
15. We say *most of* the construction of the conceptual model is part of the conceptualization stage (Section 2.5). This suggests that some part of the construction conceptual model is part of an other stage of the process model of Section 1.4. Which of the 4 steps is that, and which stage does it belong to?
16. We explain the 4 steps of constructing a conceptual model in Section 2.5. In step 3, we establish the types of properties. We see a number of different ways value sets are bound to properties. List them, and explain their differences.
17. Consider the relation `rel(a(3),b(n))`. What do the symbols in brackets mean?
18. What is an entity-relation graph? Give a reason why this name is wrong.
19. What is the difference between a quantity and a property?
20. What do we mean by the type of a quantity?

21. What is the difference between things, occurring in formal expressions, written in `this font` and in *this font*?
22. What is a compound type?
23. Why is the square of the perimeter in a rectangle at least 8 times the area?
24. What is the difference between an ordinal scale, an interval scale and a ratio scale?
25. What is an equivalence class?
26. What is the meaning of $p_{1;3}$ and $p_{3;2}$ in Expression 2.5?
27. We regularly encounter factors such as $p_{m;U}$. What is the meaning of U in these expressions?
28. Explain in your own words what a dimension is, try to avoid the word 'equivalence class'.

2.12 Exercises

1. 'Eiffeltower' is a concept. Give a bundle of properties, defining 'Eiffeltower' such that it is a singleton, and give a bundle properties such that it is a set with multiple elements.
2. In the street lamp example, given the problem and the purpose, add at least two concepts.
3. In the street lamp example, for each of the concepts, either
   - add one or more meaningful properties, given the purpose, or
   - give an argument why no more properties are necessary.
4. In the street lamp example, give at least 5 relations between properties.
5. In the street lamp example, an important quantity is the distance between adjacent street lamps. This quantity could be a property of two concepts in our model. Which two? Give advantages and disadvantages of both choices.
6. In the street lamp example, there is a 3-ary relation, `sees`. Answer the following questions. Hint: think of a street in the rain, and and compare this with a dirt road ▷60 .
   (a) Explain exactly what the meaning of `sees` is. What should this relation calculate?
   (b) In the current version of the conceptual model, there are not enough properties to fully define `sees`. Name some of the properties we additionally need.
   (c) When is it possible to replace `sees` by two other relations that are both 2-ary?
   (d) When is it not possible to replace `sees` by other relations that are both 2-ary?
7. Give at least 4 examples of relations (not necessarily restricted to the street lamp casus) that are 3-ary, where two can be replaced by two 2-ary relations, and the other two cannot.
8. Similar to the street lamp casus, build a conceptual model for supporting the decision whether or not the owner of a private house should have solar panels installed. Go over all four steps in the construction of the conceptual model.

9. Similar to the street lamp casus, build a conceptual model for verifying if a governmental health agency has sufficient medicine in stock to remedy the outbreak of a contagious viral infection. Go over all four steps in the construction of the conceptual model.

10. Similar to the street lamp casus, build a conceptual model to help planning the supply of fresh vegetables to a super market. Go over all four steps in the construction of the conceptual model.

11. We discuss three types of totally ordered scales: a ratio scale, an interval scale, and a scale (such as Mohs' scale) which is not even interval scale. Give an example of a problem that can be approached by Mohs-type scale(s).

12. Give three examples of ratio scales, at most one of them from physics, and give an argument why they are ratio scales.

13. (*) Proposition: 'a ratio scale must consist of rational numbers or real numbers'. Give arguments in favor and against this proposition.

14. We derive the formula for the oscillation time of a pendulum from the dimensions. Do the same for a mass-spring system.

15. The lens-makers formula (if necessary, consult Wikipedia) cannot be derived using dimensional synthesis. Why not? What does this example teach you about the usefulness of dimensional analysis for deriving formulas?

16. You give a pizza-party: you invite N people. Everybody consumes S slices of pizza. A complete pizza contains P slices. The price for a pizza is E. Using dimensional synthesis, give an expression of the amount of money every guest has to pay.

17. A farmer possesses chicken; these lay eggs and consume chicken food. Which quantities are needed to decide whether this farmer should purchase an additional chicken? Using dimensional analysis, derive a formula to answer the question whether or not the farmer should purchase another chicken.

18. Suggest a problem that can be solved in the same way as Problem 17, and solve it.

Exercises concerning Mathematical Tools (Section 2.8)
Locate all critical points and classify them using ACCEL for

1. $f(x, y) = x^3 - 3xy + y^3$, follow this link.

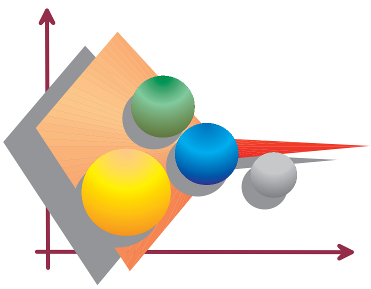
2. $f(x, y) = 2x^2 + y^3 - x^2y - 3y$, follow this link.

3. $f(x, y) = x^2 e^{-x^2-y^2}$, follow this link

Adams: §13.1: 7, 13, 19, 22; §13.2: 1, 3, 5; §13.3: 4, 16.
Smith and Minton: §12.7: 31, 35, 49, 50; §12.8: 9, 12, 15, 30, 47, 59, 60.

Chapter 3

Time for Change



'Now is past for the future and future for the past'

You see an empty green rectangle. Suddenly, from the left enters a white, shiny spherical object, followed by a red, similarly shaped one from the right. They get closer and closer, and then they collide. Their routes have drastically changed: the white ball leaves the scene at the top, whereas the red ball vanishes in downward direction. Next we see the same movie again, now played in reverse. The white ball comes in from above, the red one from below and after interaction they leave in horizontally opposite directions. We witness an equally plausible rendition of two colliding billiard balls. Apparently, for physical processes such as simple collisions between point masses or rigid spheres, the direction of time is irrelevant. Only when we look at a larger scale, say, of a complete carambole, there is a difference between past and future. The hit with the cue comes first, initiating the first ball's movement, and friction and collision losses gently slow everything down until the balls come to a standstill after a while. Time reversal at this scale would cause motionless balls gently to acquire speed, until they miraculously bump against a cue, held at exactly the right place by the billiard player, who then plans the shot ... which is clearly in conflict with our daily experience.

3.1 Change needs Time

Physical time at the microscale is REVERSIBLE; this is not true if friction or other complex processes come into play. We say that time is MICRO-REVERSIBLE, and MACRO-IRREVERSIBLE. Physical time comes in a symmetric and a non-symmetric version. The same is true for human perception. To some extent, we can anticipate the future. Sometimes, we know what will happen, which enables us, for instance, to catch a ball. In most cases, however, the difference between past and future is obvious: we sense the difference between remembering and anticipating[1].

Both in physics and in the subjective experience, time often has a direction, sometimes called the *arrow of time*. The arrow of time relates to *cause* and *effect*. Causes, effects and the advance of time together form the ingredients of PROCESSES. A *process* is something that involves change over time. Events in processes may be linked by cause and effect-relations. An effect can never precede its cause, but not in every pair of events where one precedes the other, the first causes the second.

3.2 Introduction to Processes

In a process, things sometimes happen one after the other (I switch on the radio; next I switch on the light); sometimes one is a result of another (I switch on the light, so the room gets illuminated), sometimes they are unrelated (I sneeze, and somebody else switches on the light). A process may have conditional steps (if the sun shines brightly, I may decide not to switch on the light), and some conditions are purely time dependent (it is November, and it is four o'clock, so I switch on the light). Sometimes things need to be repeated (the radio is old, so I need to repeatedly slap until it starts working). Finally, sometimes we need to wait for things, either for a known amount of time (I wait until 20:00 to hear the news on the radio), or for an unknown amount of time (I wait to go out until the postman arrives).

All these sorts of things need to be considered if we want to describe a process.

3.2.1 States and State Charts

By means of a conceptual model, as described in Chapter 2, we describe the state of affairs of a modeled system in terms of concepts, properties, values and relations. This is fine as long as nothing changes. If something does change, we could start again and set up a conceptual model

[1] The image of a snail has been taken from http://www.rgbstock.nl/photo/mhXQ074/Slak+2

for the situation after the change, but this is obviously impractical in the case of many changes. We need another device to denote processes, involving changes in a modeled system.

We will use the idea of a so-called state chart to do so. A STATE CHART is a graphical means to denote things that take place over time. A state chart is a collection of STATES. A state is a snapshot of a system. That is: a representation of that system, containing all its concepts, their properties and the current values of these properties. In other words: conceptual models, as encountered in Chapter 2, are descriptions of states.

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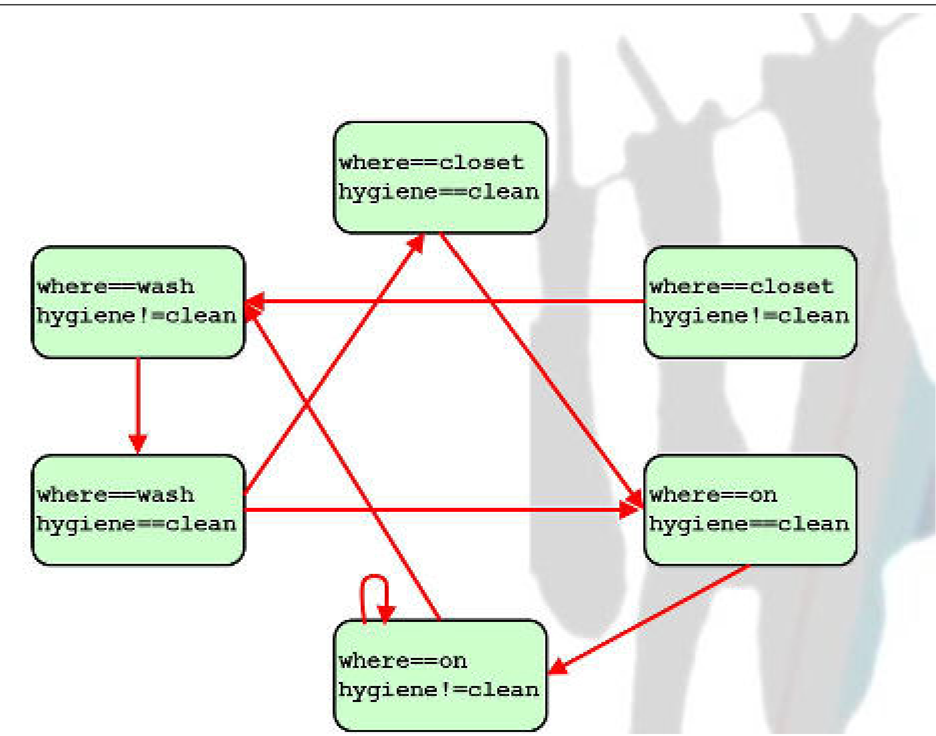


A state chart depicts the relevant transitions for a dynamical process.

Figure 3.1: A sock, described by the properties 'where' (with values 'on', 'closet', and 'wash'), and 'hygiene' (with values in between 'clean' and 'dirty'). (Illustration source: Kees van Overveld)

To illustrate 'state', consider the lifecycle of a sock. We characterize a sock by two properties. First, its location (called `where`) with values `closet` (stored in the closet), `on` (on a foot), and `wash` (being washed). Next its hygienic condition (called `hygiene` with a range of values, {`'clean'` ... `'dirty'`}.

States differ with respect to which values are currently assumed by the properties. This is called BINDING. 'Value v is currently bound to the property p' means that, at this time, v is the value of p. For a clean sock, laying in the closet, `where` is bound to `closet`, and `hygiene` is bound to `clean`.

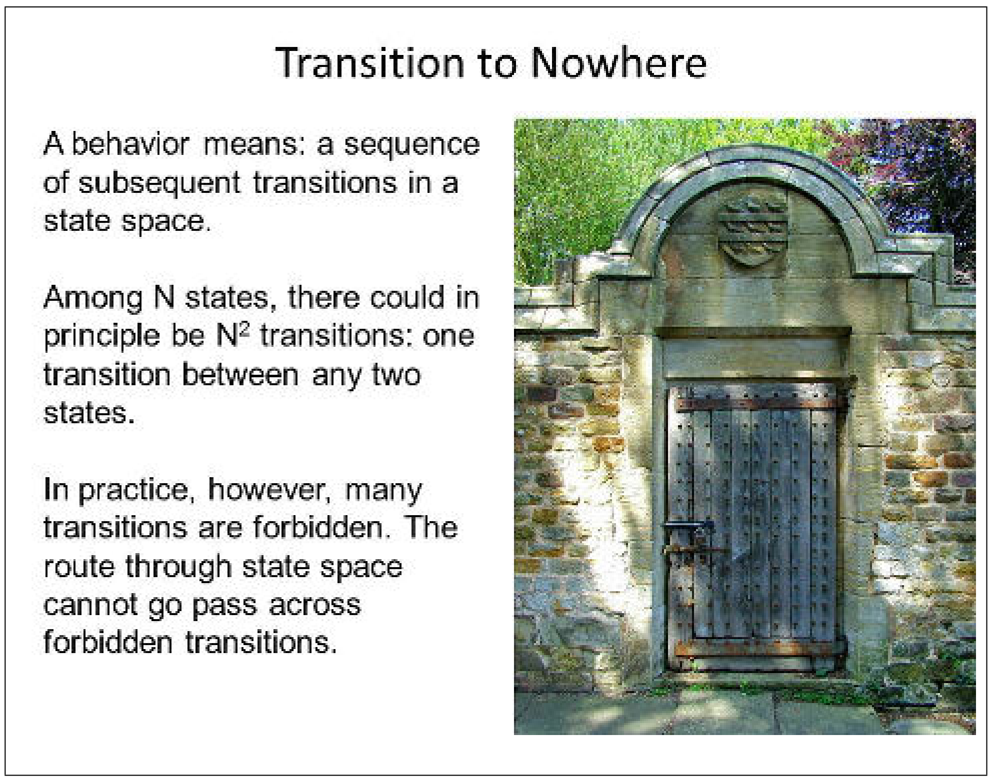
For any conceptual model there is a number of possible states. If, for the sock in our conceptual model, we only distinguish the hygiene values 'clean' or 'something else than clean', there is a total of 6 states. These correspond to all possible combinations of bindings, namely: (`where==closet, hygiene==clean`),(`where==closet, hygiene!=clean`), (`where==on, hygiene==clean`), (`where==on, hygiene!=clean`), (`where==wash, hygiene==clean`), and (`where==wash, hygiene!=clean`). We use the notation `a!=b` to say that property `a` is bound to some other value than `b`.

When a system goes from one state to another state, this corresponds to a change in binding. We call this a STATE TRANSITION [▷61], or 'transition' for short.

Everything that happens to a sock, affecting its location or its hygienic condition is a transition.

We assume transitions to take place instantaneously. In the sock example: there is one, indivisibly short instance where value `closet` for property `where` is replaced by `on`, et cetera. The fact that actually putting on a sock may take several seconds, is not accounted for in the state chart. The time *in between* subsequent transitions, however, can be indefinitely long. A state, characterized by `where==wash` may take an hour or so; a state characterized by `where==closet` may take arbitrarily long.

Some transitions may occur freely; most transitions, however, are subject to rules or conditions[2]. For instance, a transition from `hygiene==clean` to `hygiene!=clean` can only take place if `where==on`: socks in the closet don't get dirty, and socks being washed get from dirty to clean. Rules or conditions can forbid a transition to take place at all: the transition from (`where==on, hygiene!=clean`) to (`where==closet, hygiene!=clean`) is forbidden since we don't put dirty socks back into the closet.



Sometimes we may want to express that a transition takes place going from a state to *the same* state. For instance, we have a state that is defined by `where==on, hygiene!=clean`. If the sock gets dirtier, the value of `hygiene` changes, but since it was not clean before, it will stay not clean. So there is a transition where `hygiene` assumes a dirtier value, but since our choice of states only distinguishes the cases `hygiene==clean` and `hygiene!=clean`, this does not involve moving to another state. This trick helps to prevent the number of states becoming too large; in the next Section we will see that a large number of states is an often occurring problem in dealing with change in models.

State Space

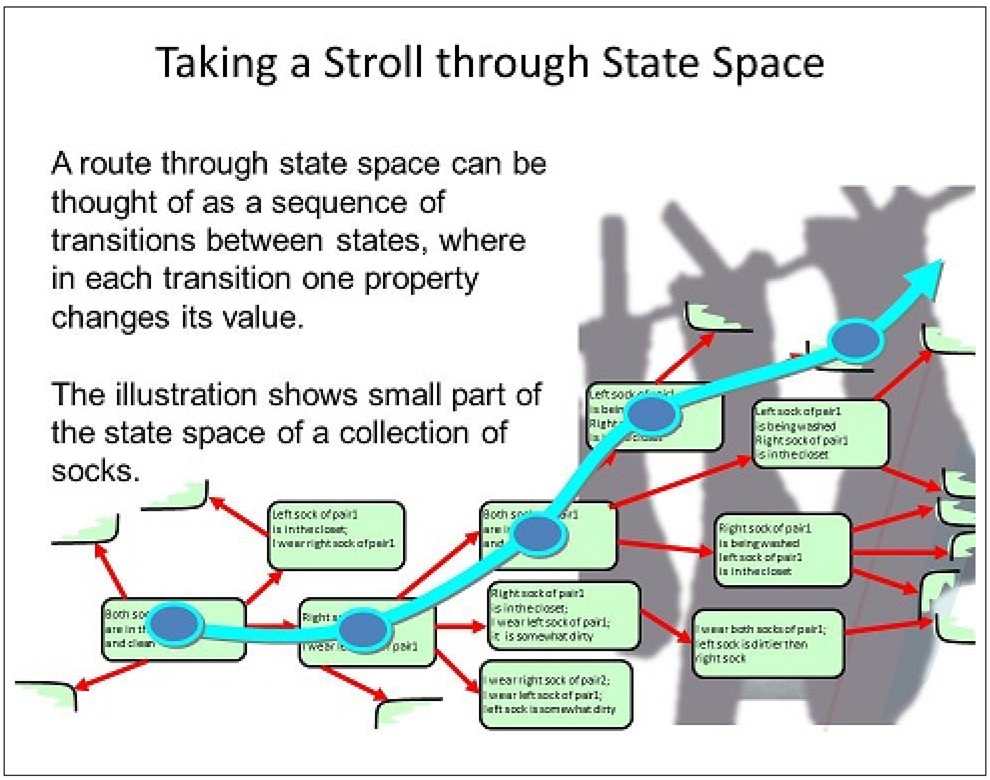
States, defined as the binding of values to properties, are numerous. Two properties, one with 3 values and the other one with 2 values give $3 \times 2 = 6$ states. Adding one further property to a concept in the modeled system multiplies the number of possible states of that system with a factor equal to the number of different values this new property can assume. E.g., if we allow

[2]The image of a closed gate is taken from http://www.rgbstock.nl/photo/mjQB46C/gesloten+hek

socks to have holes, i.e. introducing the property `nrHoles` with type $\{0, 1, 2, \mathtt{many}\}$, the number of states becomes $3 \times 2 \times 4 = 24$. If the conceptual model of a system contains N_p properties, and property i has m_i values, the number of states N_s is no less than

$$N_s = \Pi_{i=1}^{N_p} m_i. \tag{3.1}$$

All states of a system together are called the STATE SPACE. When a system develops over time, the changes it undergoes form a route through its state space, assuming one state after another. A route through state space typically visits only a limited number of states, going from some initial state to a final state. It is sometimes called a TRACE or a BEHAVIOR ▷62 .



The maximal number of possible transitions for N_s states is N_s^2, including transitions leading from a state to the same state. Transitions may be forbidden, however, yielding a number of *allowed* transitions that is often much less than N_s^2. We saw the example in the life of a sock where the `location` of a dirty sock cannot change to value `closet`. Due to this restriction, and many others, the number of transitions actually allowed is merely 8 instead of the maximum of 6×6=36. In a physical system, such as colliding billiard balls, energy conservation is an example of a restriction. It forbids that the sum of kinetic energies of the balls after the collision exceeds their sum of kinetic energies before the collision. This is also an example of a forbidden transition.

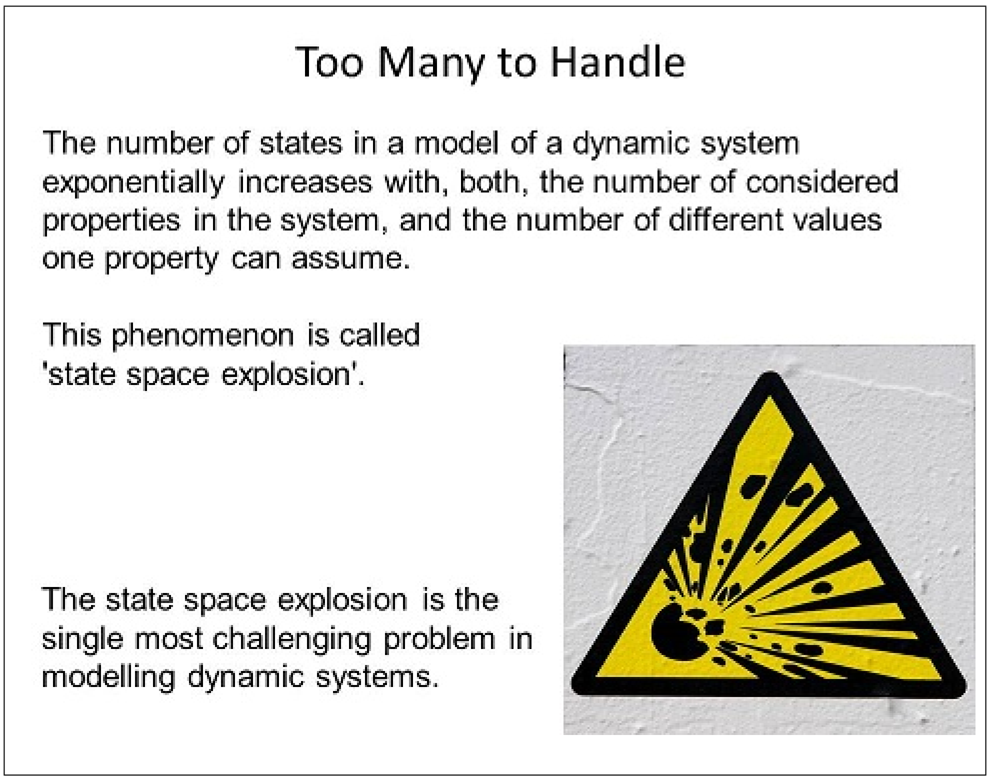
Transitions can be permitted or forbidden for various reasons: in physics constraints often are conservation laws; financial transitions may be constrained by credit limits; chemical reactions may not occur because of the absence of some reagens or catalyst, and rules of good housekeeping dictate what transitions for socks are forbidden.

Clicking this link starts the ACCEL modeling environment with a script running that allows you to interactively experiment with the state chart for the lifecycle of a sock.

Assuming that a state transition takes no time, we can define that at every transition, only one property changes its value, which makes the state chart easier to understand. In the case of a billiard ball collision: there is no difference between saying that the red ball and the white ball change their velocity at the same instant, or that there is an infinitesimal delay between the two changes ▷63 .

The size of the state space is immense; the collection of possible routes in it is even larger. Indeed, if we consider only routes with a length of N_t transitions, the number of routes, N_R is

$$
\begin{aligned}
N_R &= \Pi_{i=1}^{N_p} \Pi_{j=1}^{N_t} m_i \\
&= (\Pi_{i=1}^{N_p} m_i)^{N_t}
\end{aligned}
$$



This number grows explosively[3] both with N_p and with N_t; hence the name STATE SPACE EXPLOSION. The state space explosion is the most challenging problem in modeling dynamical systems. For any non-trivial system, it is intractable to account for all possible routes explicitly.

State Space Reduction: Symmetry

To reduce the size of the state space, SYMMETRY can sometimes be used. Symmetry is the condition that an entire system can be known even if only part of it is given. For instance: if only the left hand part of a mirror-symmetric piece of clothing is drawn, a capable tailor can make the entire piece. This is an example of SPATIAL symmetry. Temporal symmetry, for instance, applies in the example in the introduction of this chapter where the behavior of a billiard ball collision is the same when time is reversed (time reversal symmetry), or to express that the behavior of billiard balls in a carambole, say, made at noon, wouldn't be different when that carambole was made at teatime or at midnight (time shift symmetry). Temporal symmetry also applies to periodic phenomena. Knowing the motion of one swing of a friction-less pendulum is enough to know the entire behavior. Symmetries, other than spatial and temporal exist: permutation symmetry, for instance, occurs when we swap White's two bishops in a game of chess.

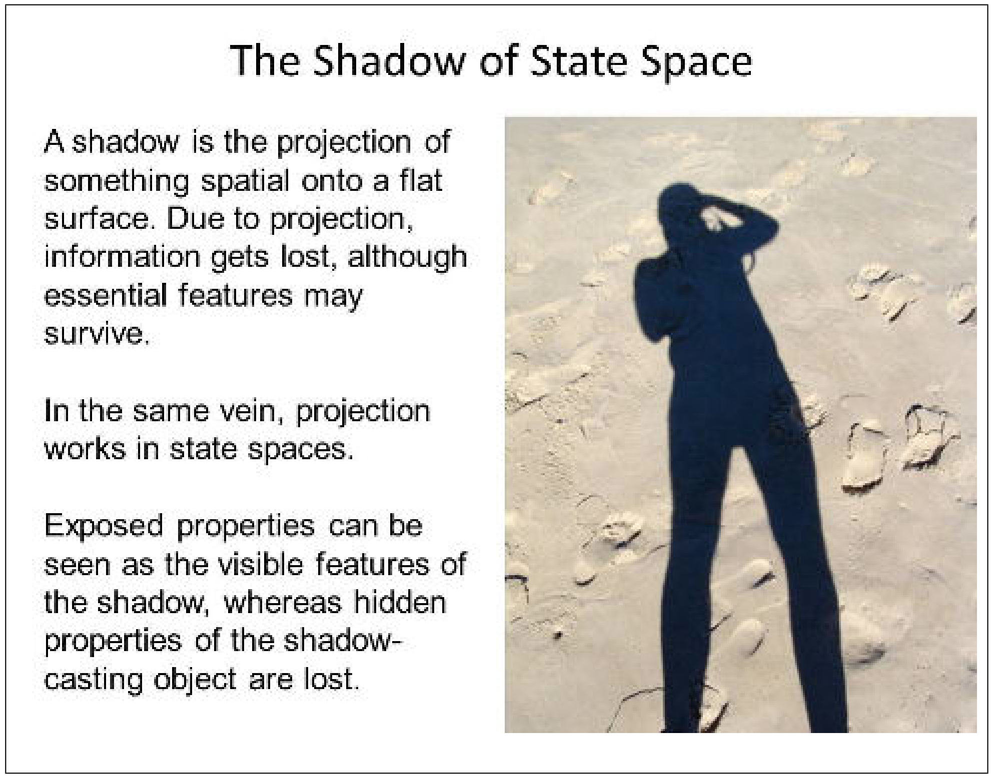
Symmetry can help to reduce the size of the state space of a system. Suppose we want to verify safety properties of the Dutch railroad signaling system. A conceptual model contains

[3] The image of an explosion hazard warning sign is taken from http://www.rgbstock.nl/photo/of8Lrbu/Gevaar

representations for all signals, all trains and all railroad switches. 'Safety' can be described in terms of states: it consists of requirements such as

- two trains shall never occupy two adjacent railway segments;
- for a switch, the signal in at most one of the branches is green;
- signals in all railroad segments leading to one with a red signal carry orange signals;
- ...

If every REACHABLE state in the state space satisfies all conditions above, the signaling system is formally safe. A *reachable* state means: a state to which a permitted transition, departing from an other reachable state, leads. There is always at least one reachable state in a dynamic process; this is called the INITIAL STATE.



Symmetry helps to reduce the state space of the railroad signaling system because the identity of trains does not matter for the verification of the signaling system (permutation symmetry). If safety is verified for one possible set of trains, we can permute these trains arbitrarily, and safety follows in the state with permuted trains as well.

State Space Reduction: Projection

PROJECTION[4] is a further means to help reduce state space. Projection means: limiting the number of properties, or the number of values for properties, considered in the model, to achieve the model's purpose with a reduced state space.

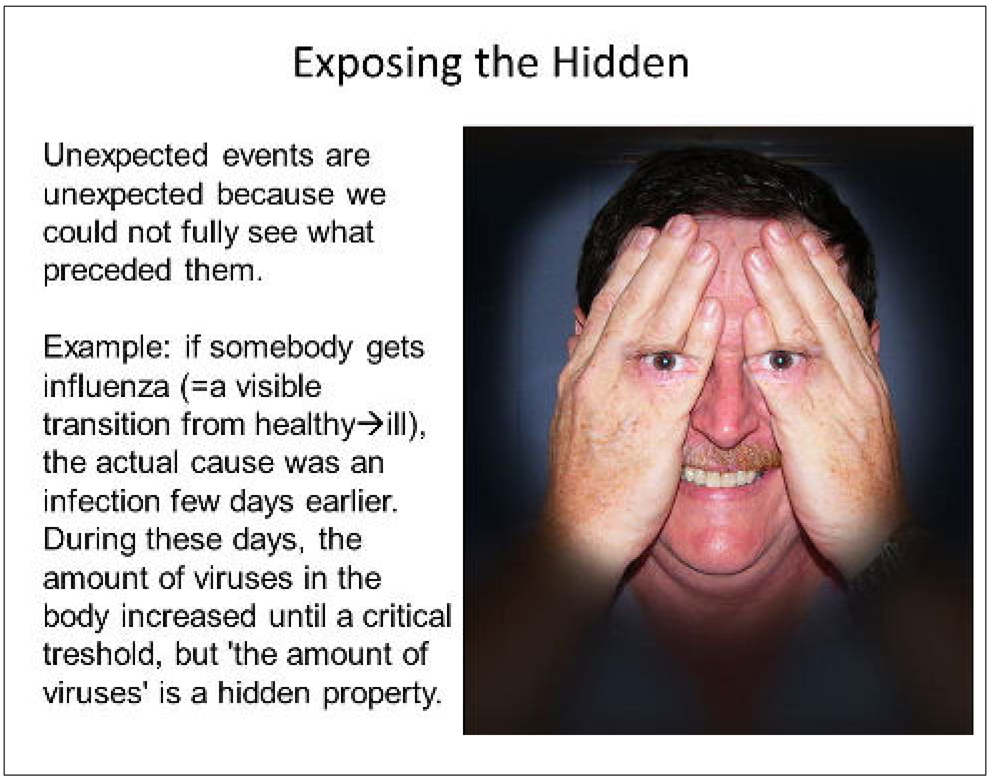
We illustrate this with an example.

In the lifecycle of a sock, there are many properties that could be taken into account. Apart from its location and its hygienic condition, we could keep track of its color (`color`), and the number of holes (`nrHoles`). The value of `color`, however takes a value that does not change over time. There are no transitions having effect on color, and therefore ignoring the property `color` from the state has no effect for the state space. For `nrHoles` this is different. Due to wear, `nrHoles` may

[4] The image of a shadow is taken from http://www.rgbstock.nl/photo/mmeHudS/%3E+self-portrait

increase, and due to repair it may decrease. Hence there are two transitions that have effect on `nrHoles`, and the total number of states of a sock is multiplied by the maximal number of holes we want to consider, according to Expression 3.1. Given the purpose of the model, it may be safe to ignore the occurrence of holes, yielding a considerably smaller state space. Also, we see that the size of the state space is determined by the number of values in {`'clean'... 'dirty'`}. If we, for instance, distinguish only 5 levels of dirtiness instead of 10, the number of states halves, and the number of transitions reduces roughly by a factor of 4 .

Exposed and Hidden Properties



This suggests the idea of EXPOSED PROPERTIES and HIDDEN PROPERTIES[5]. The exposed properties together determine the state transitions, observable from the 'outside' of the system. Changes in the values of hidden quantities go unnoticed. Hiding properties lowers the number of perceivable transitions.

Projecting can be done by leaving out quantities ▷64 , such as `nrHoles` in the sock example.

As a second example: remove the seconds-hand from an analog clock, and the passing of seconds no longer leads to visible (exposed) transitions. The state of the clock is projected down from three quantities (hours, minutes, seconds) to the two quantities (hours,minutes). The inner (hidden) states of the clock, however, still change at least every second. This projection reduces the state space of the clock from $12 \times 60 \times 60 = 43200$ states to mere $12 \times 60 = 720$ states.

Projecting may mean, however, that the modeled system can no longer be fully understood. In the clock example: if we can't inspect the state of the second hand, any transition of the minute hand comes as a surprise.

So there is a trade-off: having many exposed properties gives a large state space; having few exposed properties gives a smaller state space, but the model may become incapable to explain all transitions.

[5]The image of a hidden-exposed face was taken from http://www.rgbstock.nl/photo/nbtJ04e/Ik+zie+je+nog+steeds+...+2

Projecting may mean: hiding *properties*. More often, it means: hiding *values*. In the sock example, we ignore all grades of dirtiness other than `clean`. We only distinguish `clean` and *not*`clean`. Obviously, this is done with the goal of achieving a smaller state space. So: the number of states of the modeled system is often huge, and the number of transitions therefore even much more so, but by clever choices for the hidden and exposed properties and values, the actual *purpose* of a model may be achieved with considerably fewer states.

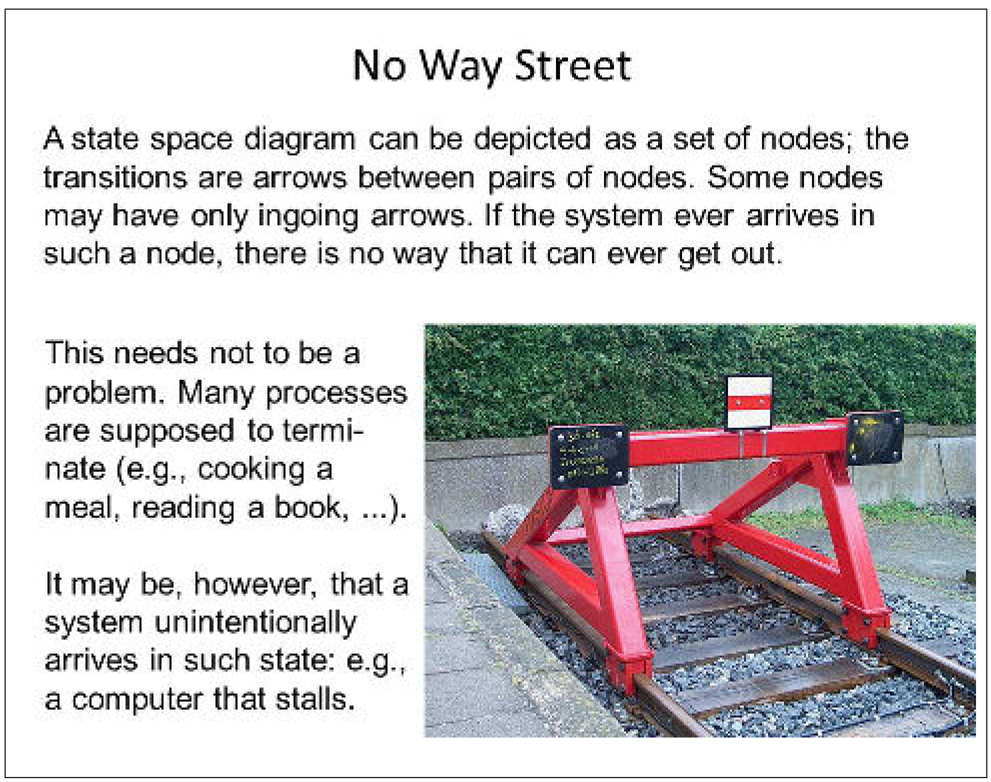
Projecting, with the purpose of the model in mind, is a powerful device to mitigate the state space explosion that would result from inadvertently adding more properties to the conceptual model.

There is no immediate right or wrong with respect to projection. The purpose of the model dictates which exposed behavior, and therefore which exposed properties, we need.

Hierarchy and Orthogonality

State charts have been developed over the past decades into a powerful device for modeling dynamic systems, in particular by adding hierarchy. Hierarchy is a way to hide and expose properties, so that the internals of a dynamical system can be modeled without state space explosion. Also so called ORTHOGONAL or independent subspaces as part of present day state chart concepts helps to mitigate state space explosion.

A formal treatment of these more advanced state chart concepts falls beyond the scope of these notes.



3.2.2 Applying State Charts

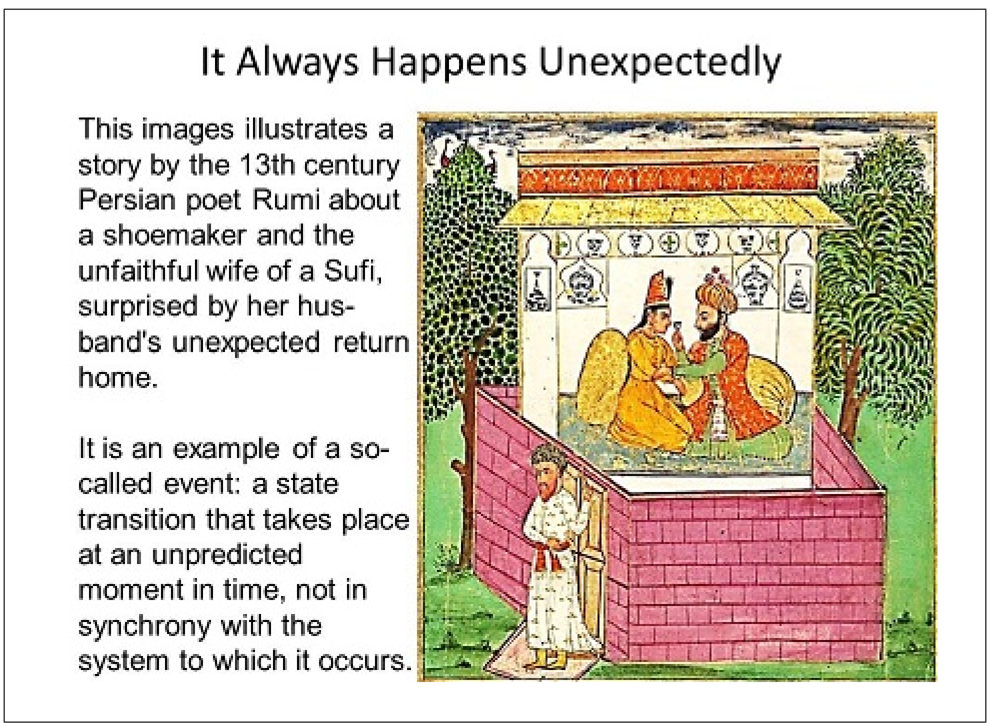
We mention a number of applications of state charts:

- verify if there are no DEADLOCKS[6] (=dead end states). A dead end state is a state that has no outgoing transitions. In such a state, the system has no way to make progress. An example from the railroad signalling case: two halted trains, waiting for each other's departure. A related anomaly is LIFELOCK. Then there is a small collection of permitted states, but none of these states has a transition

6 The image of a buffer stop is taken from http://commons.wikimedia.org/wiki/File:Stootblok_(staal).jpg?uselang=nl

to any state outside the collection. An example is after you-after you-blocking, occurring if two polite people both want to give way in a narrow passage;

- verify if states that should be reached can be reached. For instance, in a maintenance schedule (e.g., periodic control), it may be acceptable that, due to some high-priority exception, an occasional round of maintenance is skipped. If such interrupts occur too often, however, it could cause the complete maintenance scheme to break down. It may be necessary to verify that despite interrupts, maintenance at least takes place every once so often;
- verify if states that should be reached in some order are reached in that order. For instance, communication protocols as in computer networks (Internet, money transfer) need to be robust against network failures, out-of-order messages and perhaps against malicious attacks of the communication partner. A network failure or malicious attack is generally unpredictable, so the unwanted state transitions happen at unexpected instances, amidst the planned protocol;
- verify if occurring transitions are expected or admitted while monitoring a system;
- verify that *eventually* the behavior of a system will have certain properties. As an example, for a model for the Dutch railroad switching and signalling system: no train shall be held up *for ever* waiting for a red signal;



- argue about the synchronisation[7] of events. As follows. Things sometimes happen independent of anything else. A poor dancer may move his feet in a way that is not at all connected to the rhythm of the music. Something happening independently from the flow of events in some process P is called ASYNCHRONOUS with P. The opposite of asynchronous is SYNCHRONOUS. Synchronization means that the time order of events in one process is connected to the time order of events in another process. Example: when preparing a sandwich, the butter should be applied in between slicing the bread and putting on the topping. Applying butter is to be synchronized with the other two stages of the process.

As an example of the success of advanced use of projection: an automated parking garage in 's

[7] The illustration from the Rumi story was taken from http://commons.wikimedia.org/wiki/File:Jalal_al-Din_Rumi,_Maulana_-_A_Shoemaker_and_the_Unfaithful_Wife_of_a_Sufi_Surprised_by_her_Husband%27s_Unexpected_Return_Home_-_Image_Detail.jpg?uselang=nl

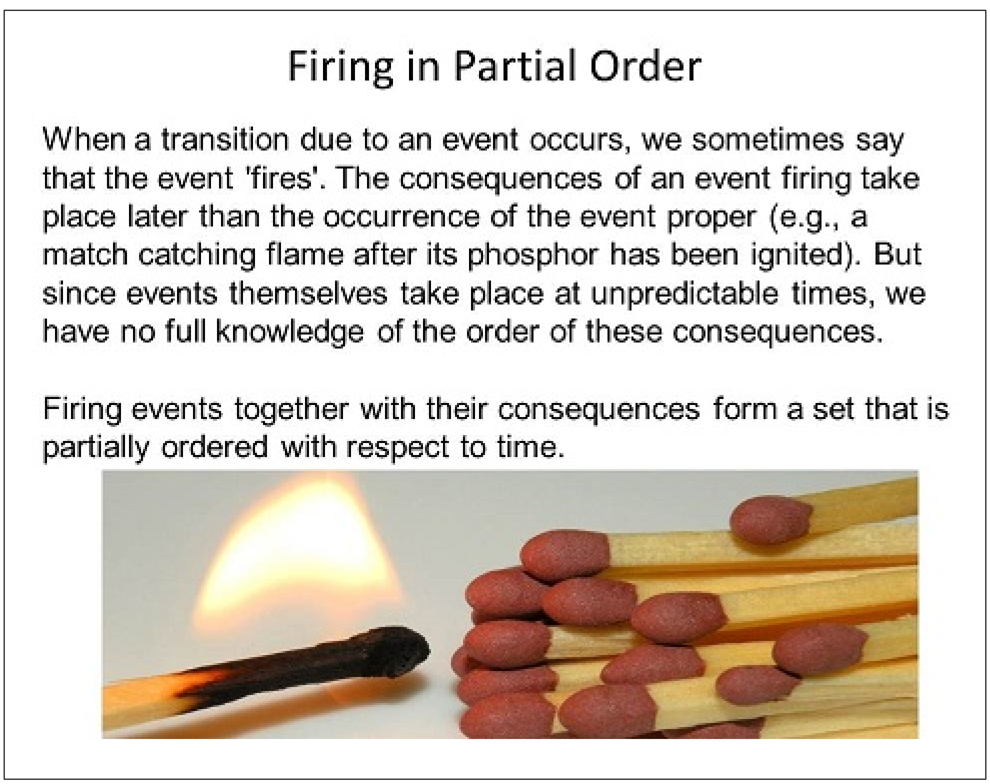
Hertogenbosch had to be verified for safe behavior (that is, no two cars should be parked in a location only large enough to hold one, etc.). A straightforward state chart model of the system amounted to some 10^{80} states, clearly beyond the capability of any computer. Clever projection helped reduce the number of states, necessary for full verification, to a mere 10^6 - which can be handled by a standard PC in a reasonable amount of time.

3.3 Time and State Transitions

We have not yet introduced the notion of time proper. In the sequel we consider three different ways of representing time in models.

3.3.1 Partially Ordered Time

Time *appears* to be totally ordered. For any two events, it appears possible to say which came first. There are exceptions, though: in the reconstruction of a crime it may be difficult to assess whether the prime suspect appeared at the crime scene before or after the fatal blow on the victim's head took place, and the distinction between these two may make the difference between imprisonment or acquittal on the ground of lacking evidence. A police inspector's report of the crime can be seen as a model with the purpose of documentation of the crime. In this model, the time order of the suspect's appearance at the crime scene and the assault could be undetermined: this is a model with *partially* ordered time.



In state charts, the time order for transitions that lead to and from some state are known[8]. Indeed, a transition leaving a state can only take place *after* the transition leading to that state has occurred. For other pairs of transitions, we don't know their order. In the sock example: we know that, with respect to the state corresponding to the sock being washed, the transition `getDirty` occurs before `getClean`, but we can't tell if the sock is put on before or after it had been stored away in the closet. In fact: the state chart describes a whole sequence of transitions, in-

[8] The photograph of a flame about to ignite matches is taken from http://www.rgbstock.nl/download/Lajla/mxJIE2k.jpg

cluding many washes, many instances of storing away a sock, and many instances of putting on that same sock.

An arrow in a state chart is a transition that can be identified with a certain point in time. The reason for a transition may be an external EVENT, that is: something that happens *outside* the modeled system which affects its state. The sock model does not represent the mechanism describing how and why socks get dirty, so the transition `getDirty` is due to an external event. Events can happen at any time, not necessarily synchronized with the process taking place in the dynamic system. We give some examples of events:

1.a telephone ringing: the system consist of a process (say, cooking), and the ring is an asynchronous interrupt that may require SERVICING, such as putting the stove off before the phone is picked up;

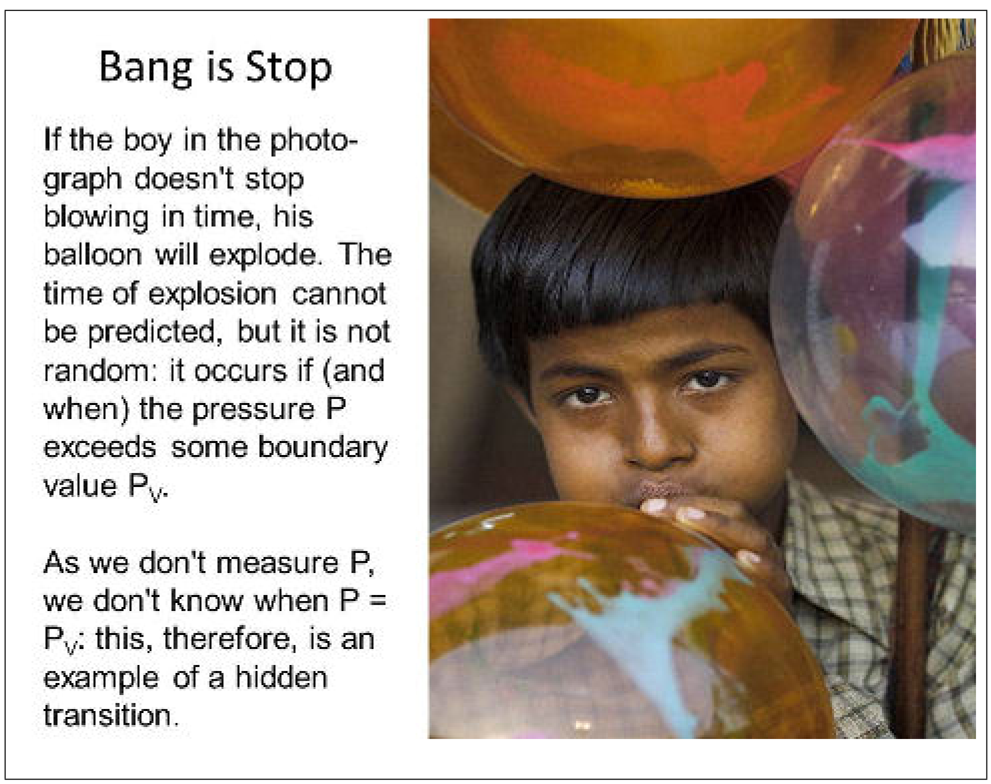
2.insertion of a coin into a coffee machine: the system consists of a coffee machine, for instance busy controlling its internal water temperature. Irrespective of the current state of the coffee machine, the inserted coins have to be registered, since the display has to show the right balance at any time;

3.colliding billiard balls: for the system consisting of a rolling billiard ball, a collision with something else is an external event.

Since events last infinitely short, we have postulated that no two events take place at the same time. Occurring events are totally ordered in time ▷65 .

Next to events, there are INTERNAL TRANSITIONS ▷66 .

Internal transitions can occur as a result of a transition of a hidden property. For example: consider inflating a balloon[9]. Exposed properties are the amount of air we put into the balloon and its volume. Hidden properties are the stress in its skin, and the maximal stress it can endure. The moment where these two are equal, the balloon explodes. This is a transition without an external event, unlike when the explosion is caused by pinching the balloon with a needle.



Again in the three examples above, internal transitions are:

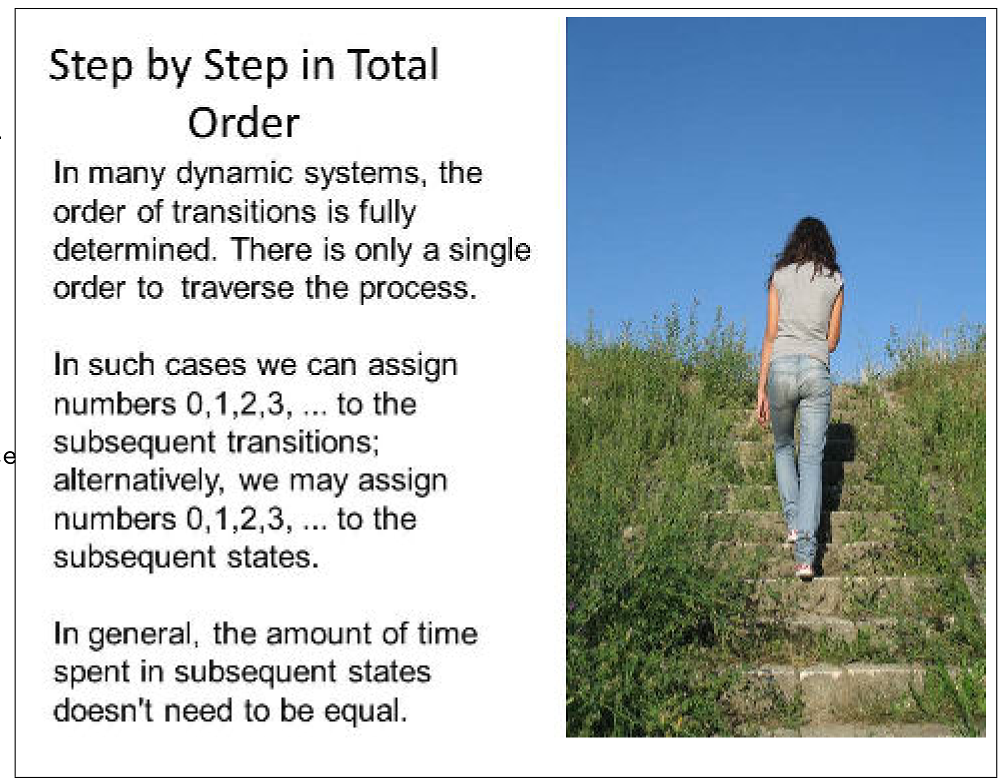
[9]The photo of a balloon-blowing boy is taken from http://upload.wikimedia.org/wikipedia/commons/a/a1/India_-_Varanasi_boy_balloon_-_2735.jpg?uselang=nl

1. the cook stirs a pan of sauce, and as a result of that, the last lump in the sauce dissolves;
2. the water inside the coffee machine is warming up to the point where the heater is switched off;
3. if locations of a rolling ball are measured with a ruler, subsequent marks on the ruler are passed in subsequent time points.

Although an actual stream of events and transitions in a process is totally ordered in time, we may only be interested in certain parts of their ordering. It may be that the model only involves events or transitions in a *partial* time order. It is also possible that the model represents *possible* transitions for the modeled system that may occur in more than one order. Then the assumption about total ordering no longer holds.

Again for the same three examples as above[10]:

1. In a telephone communication, the events `dialA` and `answersTheCallB` have a fixed order. But after the conversation, both `A` or `B` can terminate the connection. So the order of the events `terminateA` and `terminateB` is irrelevant.
2. In a coffee machine, we identify five events: `insertCoin`, `makeChoice`, `startCoffeeMaking`, `give Coffee`, and `returnChange`. There is no fixed order between `insertCoin` and `makeChoice`, and neither between `start CoffeeMaking` and `return Change`. But both `start CoffeeMaking` and `return Change` occur after both `insertCoin` and `makeChoice` have happened. A coffee machine designer may use partial ordering to specify the coffee-buying process. For instance, if `comesBefore(start CoffeeMaking, give Coffee)` and `comesBefore (giveCoffee, returnChange)` the machine may not be able to give change in a situation where coffee beans have run out.



3. Depending on the variety of billiard game, the collisions between balls and cushions may or may not be relevant: in libre style billiard, there is no order constraint; in the variety known in the Netherlands as "10 over rood", a stroke is only valid if the events `hitFirstWhiteBall`, `hitRedBall`, `hitSecondWhiteBall` take place in this order.

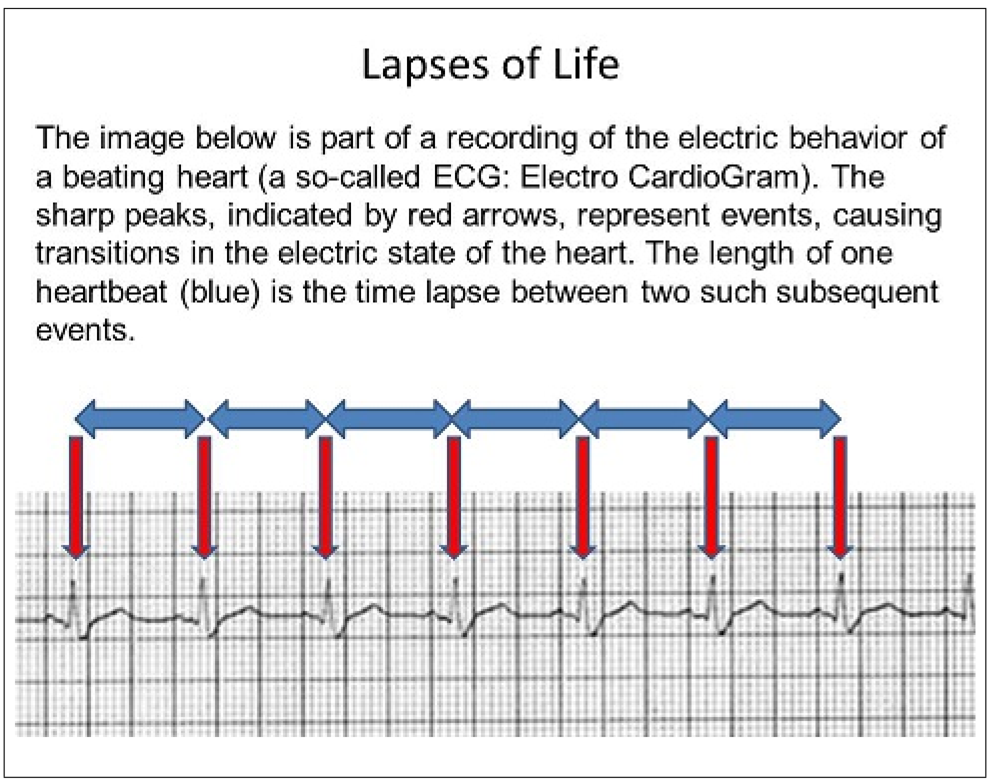
[10] The image of a lady climbing stairs is taken from http://www.rgbstock.nl/photo/mhAVAC6/de+trap+3

3.3.2 Totally Ordered Time

To properly get up a flight of stairs, one should step on the stairs one by one in the order of increasing height. The time one spends on one step, however, could very well be different from the time one spends on another. The same is true for reading a book: the pages should be turned in the order of increasing page numbers, but reading one page could take more time than reading another one.

In this section we assume total time ordering. All occurring transitions can be uniquely labeled with increasing integers, 0,1,2,3.... For any two transitions t_i, t_j the relation later(t_i,t_j) holds when $i > j$.

Time Lapses



Since we define transitions to be instantaneous, two subsequent transitions can be identified with a TIME LAPSE. A time lapse[11], Δ, is the amount of time elapsing between two subsequent transitions. Δ is a function from two transitions to $\mathbf{R}^+$, such that $\Delta(t_i, t_j) + \Delta(t_j, t_k) = \Delta(t_i, t_k)$, where $i \leq j$, $j \leq k$, and $i \leq k$. It follows that $\Delta(t_i, t_i) = 0$ for all transitions t_i. Indeed: transitions take no time.

Between t_i and t_{i+1}, nothing happens. None of the exposed quantities in the model undergoes a transition.

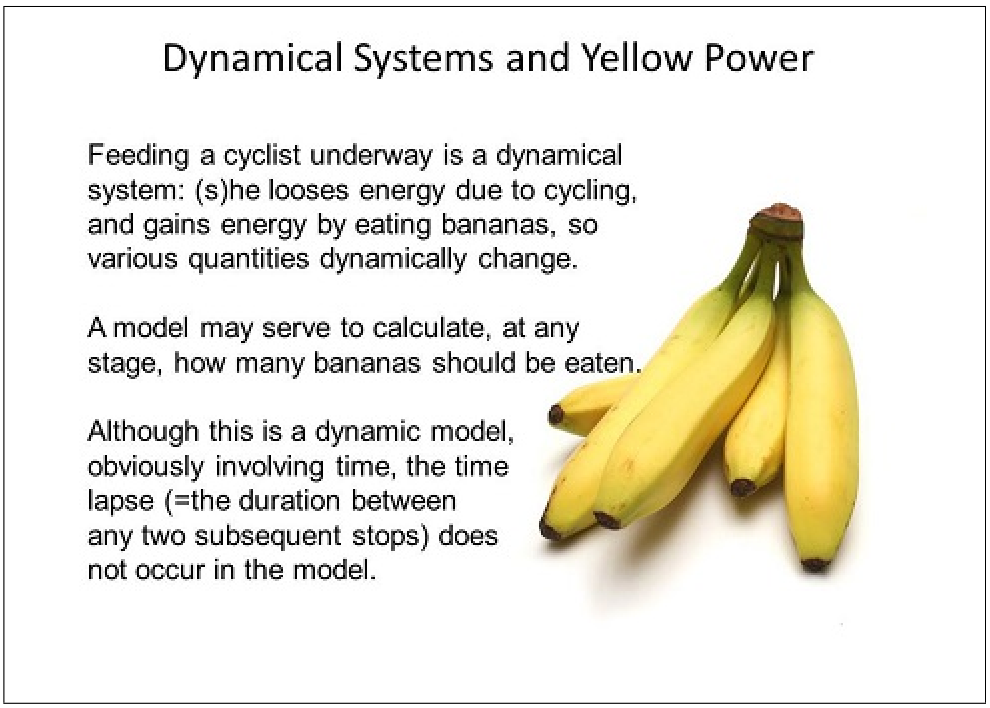
Transitions are labeled with increasing integers; states or state properties are labeled in the same way. For a state property Q, Q_i denotes its value after transition t_i. So we can also refer to state nr. i or 'state i' for short. This is the state during which Q assumes the value Q_i.

The time lapse of state i is $\Delta(t_i, t_{i+1})$, or Δ_i for short.

[11] The ECG image is taken from http://commons.wikimedia.org/wiki/File:De-ECG_RBTB_LAtrD_(CardioNetworks_ECGpedia).jpg

Total Ordering, Causality

In a model with totally ordered time, we don't require that the absolute length of a time lapse has a meaning. We give an example. We want a model to help calculate the number n_i of bananas[12], each containing B kCal, we should eat at each stop i in bicycle trip, where we have been given the amount of effort E_i, between any two subsequent stops, as a list $\{E_i\}$ with values.



We define Q_i as being the amount of kCal we have in our body at stop i. When we start the tour we have Q_0 in our body. If we don't eat any bananas, the amount of kCal at stop $i+1$ is given by $Q_{i+1} = Q_i - E_i$. If we do eat n_i bananas, however, we get $Q_{i+1} = Q_i - E_i + n_i B$. We want to choose n_i the minimal value such that $Q_{i+1} \geq 0$. Indeed, to avoid constipation, we don't want to eat too many bananas; further, we eat an integer amount of bananas. So $Q_i - E_i + n_i B \geq 0$, and $n_i B \geq E_i - Q_i$. It follows that $n_i = \max(0, \lceil (E_i - Q_i)/B \rceil)$. So $Q_{i+1} = Q_i - E_i + B \times \max(0, \lceil (E_i - Q_i)/B \rceil)$, where $\lceil ... \rceil$ means: taking the rounded-up value. So, e.g., $\lceil 4.2 \rceil = 5$. Notice that the lengths of time lapses, Δ_i, doesn't enter into the calculation of Q_i.

Causality and Functions

In the banana example, we find $Q_{i+1} = Q_i - E_i + B \times \max(0, \lceil (E_i - Q_i)/B \rceil)$, or $Q_{i+1} = F_Q(Q_i, E_i, B)$. So: there exists a FUNCTION F_Q, so that we can write a quantity Q in subsequent states as a function of earlier values of Q, and, optionally, earlier values of other quantities (such as E_i), and/or constants (such as B).

The function F_Q represents our intuition of 'causal dependency'. It expresses the causal dependency of Q_{curr} (=the current value of Q) on Q_{prev} (=the previous value of Q):

$$Q_{\text{curr}} = F_Q(Q_{\text{prev}}, P_{\text{prev}}), \tag{3.2}$$

where P_{prev} holds the values of optional other quantities at a previous time step.

In the banana example these are the E_i and B; in a more elaborate model, Q could also depend on perspiration, water intake, temperature and further quantities.

[12]The banana image is taken from http://www.rgbstock.nl/photo/mWjX0mK/Bananaaaaas%21

F_Q is a function, that is: given Q_{prev} and all P_{prev}, the value for Q_{curr} follows from evaluating F_Q. For F_Q to be a function, Q_{curr} may only depend on state quantities of *earlier* states. F_Q being a function means that there must be causal relations leading from earlier states j, with $j < i$, to state i ▷67 .

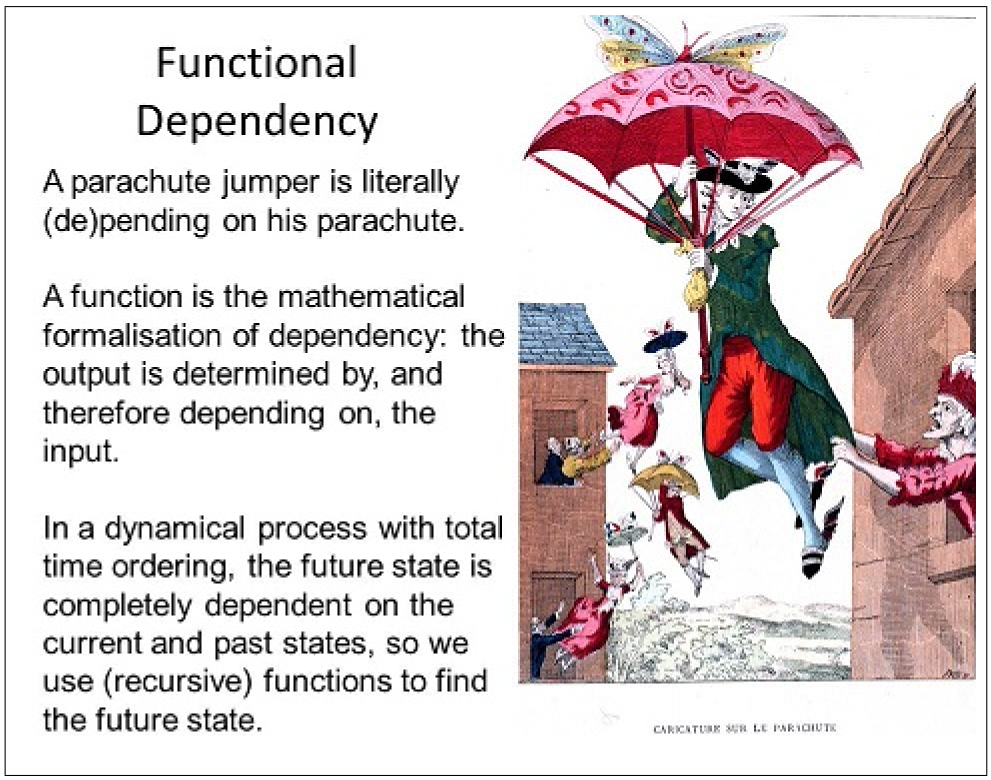
We may consider to allow:

$$Q_{\text{curr}} = F_Q(Q_{\text{prev}}, P_{\text{curr}}),$$

since this seems not to violate causality per se[13]. But suppose that we also have

$$P_{\text{curr}} = F_P(P_{\text{prev}}, Q_{\text{curr}}),$$

then F_Q and F_P *together* violate causality. Hence the more strict requirement that *all* state quantities occurring as arguments in any functions F need to be taken at an earlier point in time.



Suppose that we have no hidden quantities, and that for every quantity Q,

$$Q_{\text{curr}} = F_Q(P_{\text{prev}}),$$

is the only function in a process model. That is: Q does not depend on earlier values of itself. Then everything is static: P doesn't depend on Q, and also not on previous values of itself, so it must stay constant. Q depends functionally on P, but with unvarying P, Q also stays the same. We see that change is only possible if there is at least one quantity that depends on an earlier value of *itself*.

Evolution of Dynamical Systems

We summarize: if the purpose allows time to be totally ordered, and if we need to evaluate quantities at subsequent time points, then we could use functions F_Q such that

$$Q_i = F_Q(Q_{i-1}, P_{i-1}). \tag{3.3}$$

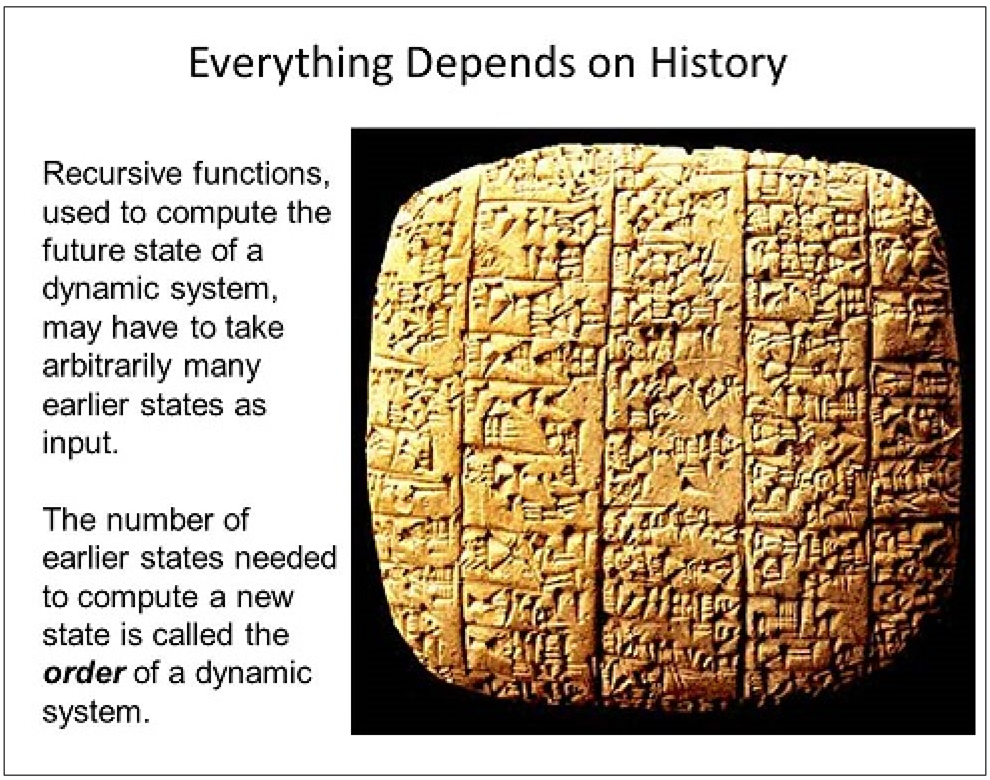
[13]The caricature on the invention of the parachute is taken from http://commons.wikimedia.org/wiki/Parachute#mediaviewer/File:Parachute_caricature.jpg

If Q_i depends on multiple earlier states,

$$Q_i = F_Q(Q_{i-1}, Q_{i-2}, P_{i-1}, P_{i-2}, \cdots), \tag{3.4}$$

involving as many earlier states as necessary.

If the oldest state that occurs among the arguments of F_Q to compute Q_i is $i-n$, n is called the ORDER of the model. For example, for a financial transaction system where a new amount of money on an account is calculated from the current amount and a transferred sum, the order is 1. For the banana example, the order was also 1; in a little while we encounter systems that are order 2.



There are systems that require large values of n: for instance[14], a protocol where a product undergoes many different transitions, and a next transition may depend on each of these. A model for a system that has memory (such as a computer, an hourglass, or a living organism) may require large n. Most mechanical and otherwise physical systems, can be modeled with n no larger than 2. We say that systems with low order are (nearly) memory-free.

If the order of a dynamical system is n, we cannot use Equation 3.4 at the first n states. We need additional information to find the values of the quantities in these first n states. These are called INITIAL VALUES ▷68 .

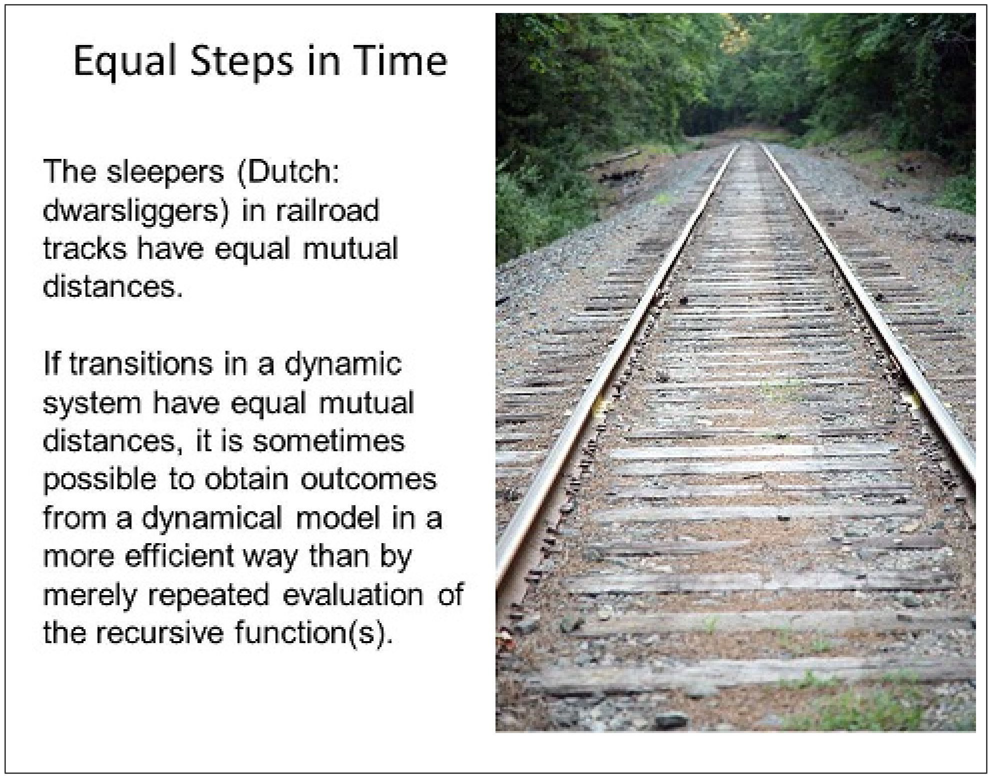
3.3.3 Totally Ordered Time; Equal Intervals

Results thus far only required totally ordered time. Time lapses between subsequent transitions could have different lengths. Next we look to the special case where time lapses are equally long.

[14]The image of a clay tablet with ancient inscription is taken from http://commons.wikimedia.org/wiki/File:Ebla_clay_tablet.jpg?uselang=nl

Equal Intervals: Periodicity

Processes often assume PERIODIC time. Consider a pendulum to measure time; it works because of its periodic behavior with clearly marked transitions, *viz.*, the two extreme positions where the pendulum reverses its direction. Our calender is based on the annual seasons, driven by the earth's orbit round the sun. Also the financial world is influenced by this rhythm, as interest on a deposit is made payable every 1st of January.



Let us study the last example in some detail[15].

At first, we ignore the periodicity. We take an example from *budgeting*: that is, planning expenses such that in the long run a sustainable financial situation occurs. There is an initial amount of money, A_0, and after an amount of time Δ_0 a fraction $s_0\Delta_0$ is spent. We expect to spend more in a longer period of time, hence the proportionality to Δ_0. We assume $0 < s_0\Delta_0 < 1$. Also we gain an amount $g_0\Delta_0$, again proportional with Δ_0. So after Δ_0, we have $A_1 = A_0(1 - s_0\Delta_0) + g_0\Delta_0$. This holds for every transition i:

$$A_{i+1} = A_i(1 - s_i\Delta_i) + g_i\Delta_i, \tag{3.5}$$
$$t_{i+1} = t_i + \Delta_i. \tag{3.6}$$

These are both in the form of Expression 3.2:

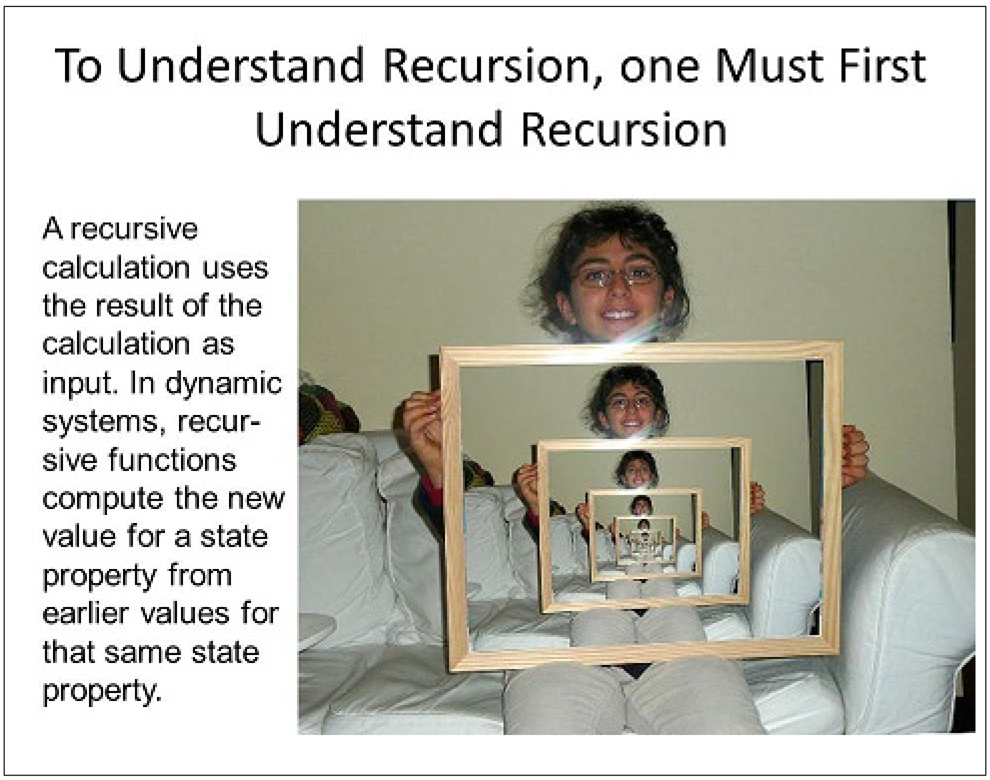
$$A_{i+1} = F_A(A_i, s_i, g_i, \Delta_i), \tag{3.7}$$
$$t_{i+1} = F_t(t_i, \Delta_i). \tag{3.8}$$

To know the amount left at time t, we seek the value for j with the largest t_j with $t_j \leq t$, since this was the transition where A reached its current value A_j. Next, just as we did with the banana example, evaluate Expression 3.7 for subsequently $i = 0$, $i = 1$, $i = 2$, $\cdots$, $i = j$. Eventually we find A_j.

[15]The image of a railroad track was taken from http://commons.wikimedia.org/wiki/File:Railroad_Tracks_In_Woods.jpg?uselang=nl

This works for *predicting* the amount left, assuming a series of expenses s_i, incomes g_i, and time intervals Δ_i. It doesn't work well for *deciding*, however. Suppose we want to decide, for a constant $g_i = g$ (e.g., salary), what expense rate $s_i = s$ we can afford for a positive result at time t. Then we can only re-calculate the entire sequence of A_i for every i until j. If we don't have any further information about s_i and g_i, there is no closed form expression for A_j when A_{i+1} is defined RECURSIVELY[16] as $A_i(1 - s_i\Delta_i) + g_i\Delta_i$.

Recursion



Recursion means: defining something in terms of earlier versions of itself. Our function F_Q in Expression 3.2 is a *recursive* manner to express the values of quantities in a dynamic system.

There is a brute force way to deal with recursion. That is to 'unroll' the sequence of applications of $Q_{i+1} = F_Q(Q_i, \cdots)$ for all subsequent i beginning with $i = 0$ ▷69 .

Evaluating subsequent states of a dynamic model by unrolling their recursive definition of Expression 3.2 is called SIMULATION. To simulate a dynamical system, we only need the F_Q's from Expression 3.2. Time intervals don't need to be equally long. With all Δ_i in Expression 3.7 different, however, we cannot do anything else but unroll the recursion. If Δ_i, s_i and g_i are constant, however, say Δ, s, g, we can compute the closed form result ▷70 which, for $i \geq 1$, turns out to be $A_i = (1 - s\Delta)^i + \frac{(1-s\Delta)^{i-1}-1}{-s\Delta} g\Delta$. This is a useful result as it tells us exactly how much we possess at any point in the future. Moreover, we can study what happens if i goes to infinity. Then $(1 - s\Delta)^i$ vanishes, and so does $(1 - s\Delta)^{i-1}$. What remains is

$$A_{\text{infinity}} = g/s. \tag{3.9}$$

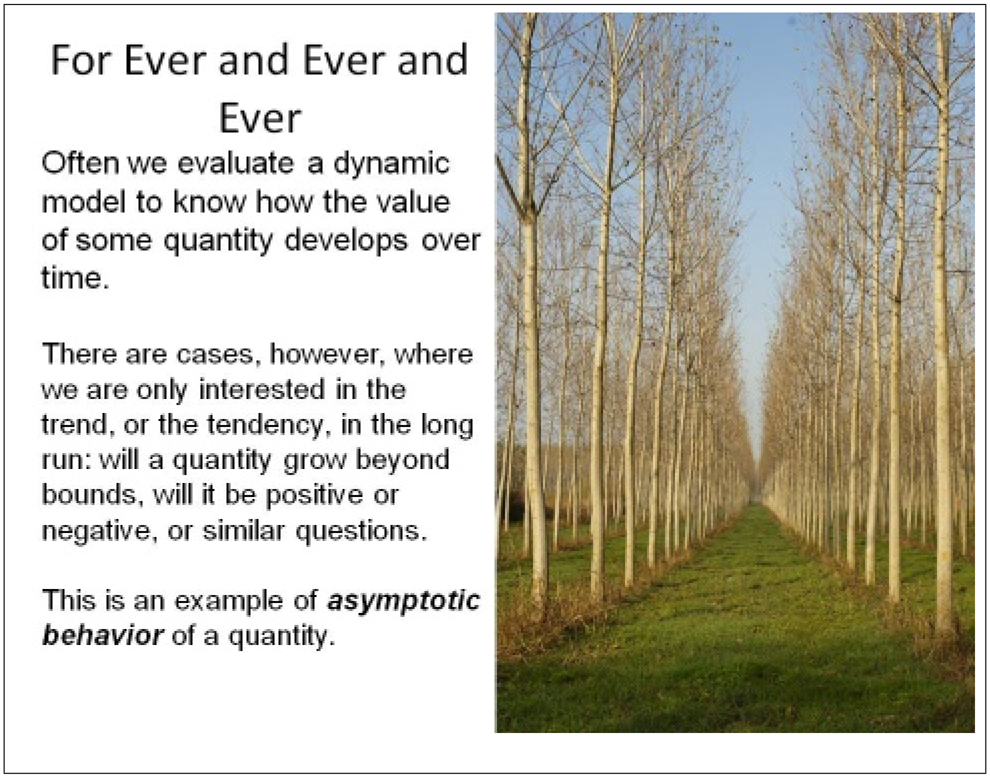
This is dimensionally consistent: g is an amount of money per unit of time, and s has the dimension of 1 over time. So g/s is indeed an amount of money. The value g/s means: if one spends, say,

[16]The photograph of a lady, holding a photograph of a lady holding a photograph of a ... is taken from http://commons.wikimedia.org/wiki/Category:Droste_effect#mediaviewer/File:Droste_1260359-nevit,_corrected.jpg

no more than 20% of one's assets per time period, in the long run ▷[71] one will own at least 5 times g over that same period, *irrespective of* Δ.

This is an example of analyzing the ASYMPTOTIC BEHAVIOR[17], that is: the behavior of the system in the long run, thanks to the closed form expression $A_{\text{infinity}} = g/s$. It helps to fulfill modeling purposes such as deciding on acceptable spending rates.

The result of Expression 3.9 shows that the actual length of the time lapse Δ is irrelevant (as long as it is constant). Whether we pay on daily, weekly or monthly base has no effect on the final situation.



This shows an interesting interpretation of our model. We can regard A as some time-dependent value, whose behavior is represented by a series A_i. We can, at any time t_t, ask what A is, just by looking up the most recent transition j and realizing that A at t_t equals A_j.

Time Lapses with Equal Length; Sampling

The values A_i, resulting from a recursive definition such as above, SAMPLE A as a function of time.

Sampling means: capturing the characteristics of a large set by looking at a small number of elements of this set. For instance, estimating the quality of a batch of oranges by testing the quality of a randomly selected handful (stochastic sampling), or recording an audio signal by means of 44100 distinct values taken per second (periodic sampling), used for recording audio on CD.

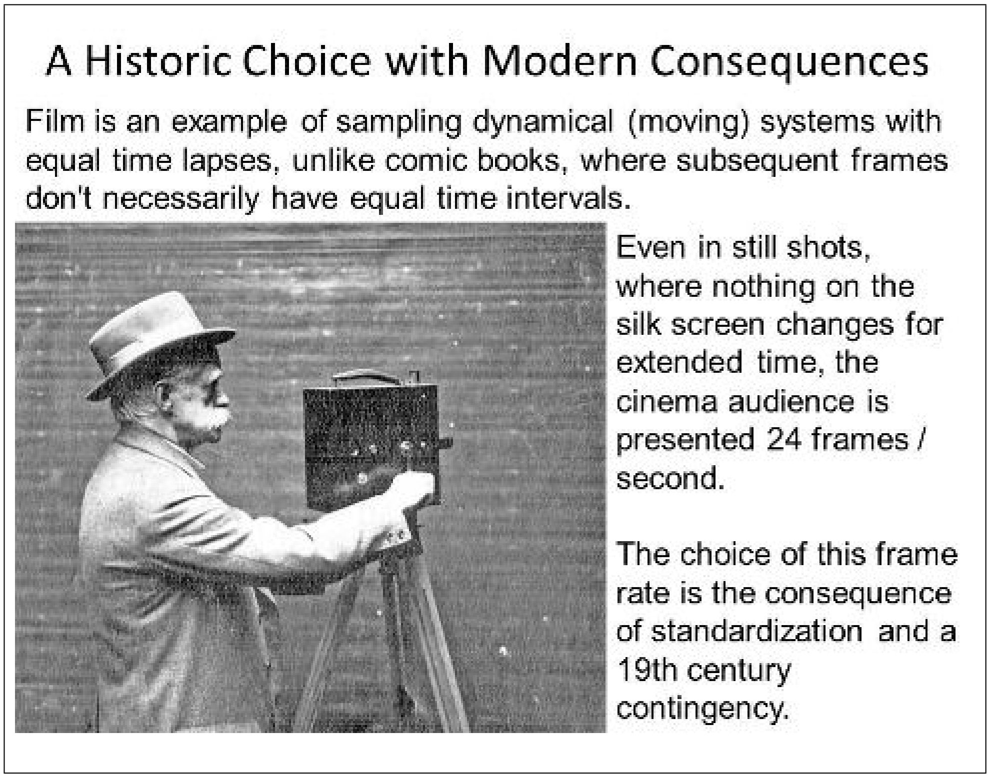
Developing a model with totally ordered time with constant Δ means *sampling* the modeled system. Unlike in the banana example, the length of the sampling intervals Δ is usually relevant for the model outcome. Let us study the example of throwing a ball.

A ball is considered as a point mass: rotation is ignored. At time $t_i = \Delta i$ it has location r_i and velocity v_i. First assume no forces. Then velocity is constant: $v_i = v$. We propose a recursive model, with uniform velocity implying that equal lengths of distance are covered in equal time

[17]The photograph of a road lines with trees is taken from http://www.rgbstock.nl/download/zatrokz/meSmz1c.jpg

lapses:

$$r_{i+1} = F_r(r_i) = r_i + v\Delta. \tag{3.10}$$



Comparing the sampled[18] locations r_i with the locations following from the secondary school physics formula, $r(t) = r_0 + vt$, we see full agreement.
With $r_i = r(i\Delta)$ we get

$$\begin{aligned} r_{i+1} &= r_i + v\Delta \\ &= r(i\Delta) + v\Delta \\ &= r_0 + vi\Delta + v\Delta \\ &= r_0 + v(i+1)\Delta \\ &= r((i+1)\Delta). \end{aligned} \tag{3.11}$$

So irrespective of Δ, sampled locations r_i agree with true locations $r(i\Delta)$.
There is another recursive definition with the same result:

$$\begin{aligned} r_{i+1} &= F_r(r_i, r_i - 1) \\ &= 2r_i - r_{i-1}. \end{aligned} \tag{3.12}$$

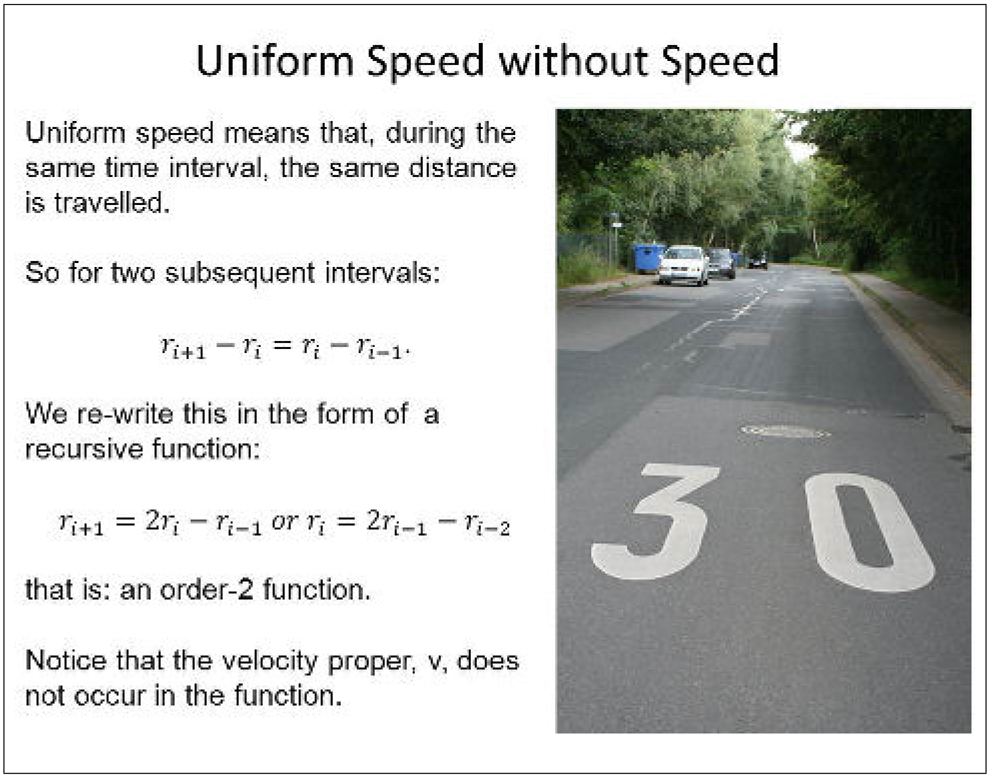
This may be understood better if we re-write it: $r_{i+1} = r_i + r_i - r_{i-1}$, so

$$r_{i+1} - r_i = r_i - r_{i-1}. \tag{3.13}$$

[18]The photograph of film pioneer Frank Mothershaw was taken from http://commons.wikimedia.org/wiki/File:FrankMottershaw-highresolution.jpg?uselang=nl

This means: the displacement between i and $i+1$ is the same as between $i-1$ and i, which is an alternative way of defining uniform speed.

So we have both an order-1 version in Expression (3.10) and an order-2 version in Expression (3.12) of a recurrent model for uniform velocity. Expression 3.12 is interesting in that it does not contain the velocity: it holds for *any* (constant) velocity[19].



Next we apply a constant acceleration a, corresponding to a force ma where m is the mass. In the absence of force, a material object moves with uniform velocity, as in Expression 3.12. In the presence of force, speed changes over time. The change of speed over a time interval is larger if the force is larger. The change of speed over time is *proportional* to the force and proportional to the time during which this force works.

From secondary school physics, the result should be:

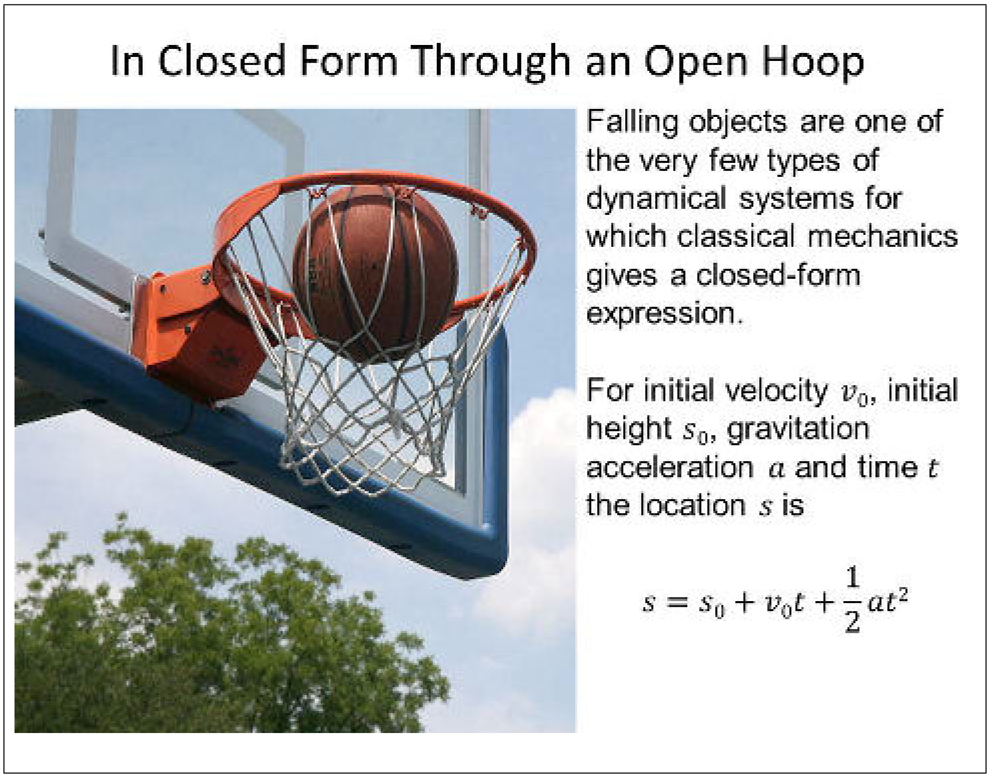
$$r(t) = r_0 + v_0 t + \frac{1}{2} a t^2 \tag{3.14}$$

By analogy with Expression 3.11, the sampled version could be tried as

$$\begin{aligned} r_{i+1} &= F_r(r_i, v_i) \\ &= r_i + v_i \Delta; && (3.15) \\ v_{i+1} &= F_v(v_i) \\ &= v_i + a\Delta. && (3.16) \end{aligned}$$

The solution for v_i from Expression 3.16 can be written down immediately, as this has the same form as Expression 3.11:

[19] The image of a road with speed limit is taken from http://www.rgbstock.nl/photo/mC2ICWS/30



$$v_i = v_0 + ia\Delta, \tag{3.17}$$

because $v(t) = v_0 + at$. So we can take Expression 3.15 and Expression 3.16 together:

$$r_{i+1} = r_i + (v_0 + ia\Delta)\Delta. \tag{3.18}$$

This is a so-called arithmetic series, of the form $r_{i+1} = r_i + x + yi$. It has a closed form solution:

$$r_i = r_0 + v_0\Delta i + \frac{1}{2}a(\Delta i)^2 - \frac{1}{2}a\Delta^2 i, \tag{3.19}$$

which can be verified by subtracting the expressions for r_{i+1} and r_i. With $t_i = \Delta i$, we find

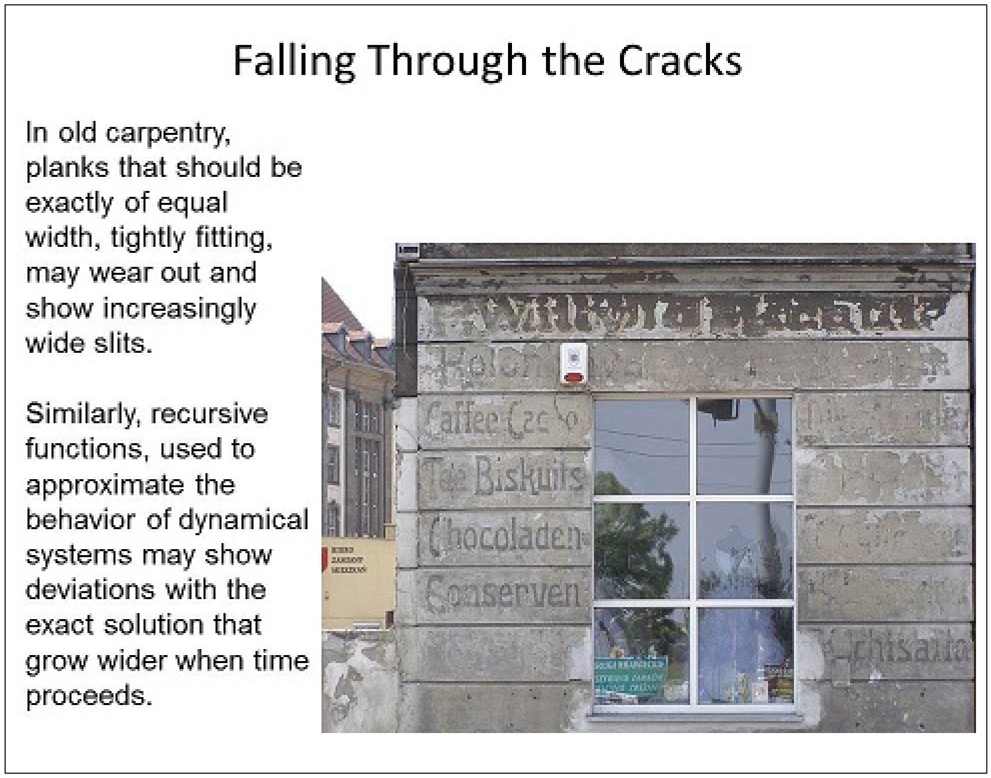
$$r_i = r_0 + v_0 t + \frac{1}{2}at^2 - \frac{1}{2}a\Delta t. \tag{3.20}$$

The first three terms in the right hand part agree with the secondary school result (Expression 3.14). The last term, however, is wrong. Our simulation yields an error term $\frac{1}{2}a\Delta t$. This error increases proportionally with time. Moreover, the error is larger when Δ is larger.

This is a very common behavior for unrolling recursive functions that express model quantities in terms of their previous values[20]. For an accurate simulation, Δ should be sufficiently small, and the approximation deviates more when time proceeds. In Expression 3.20, halving Δ halves the error for any t. We should realize, however, that halving Δ also obliges us to do twice as many caluclations in unrolling Expression 3.15 and Expression 3.16 to arrive at r for a given time point t. There is a trade off between accuracy and time needed for the calculations. This is very common in simulations of all kinds.

[20]The image of a falling ball is taken from http://commons.wikimedia.org/wiki/Category:Falling#mediaviewer/File:2011-06-07_Basketball_in_hoop_still_shot.jpg

Halving Δ in order to halve the error makes unrolling these recursive functions a so called 1ST ORDER APPROXIMATION. There are better schemes where halving the time step gives a reduction of 4 (so called 2nd order approximation) or even 8 (third order approximation)[21].



For a point mass, subject to a constant force, we can easily fix the flawed recursive definition from Expression 3.15. There is a recursion relation that gives the exact motion of a uniformly accelerated point mass. We set

$$\begin{aligned} r_{i+1} &= F_r(r_i, v_i) \\ &= r_i + v_i\Delta + \frac{1}{2}a\Delta^2; \\ v_{i+1} &= F_v(v_i) \\ &= v_i + a\Delta. \end{aligned} \quad (3.21)$$

Notice that now the difference $r_{i+1} - r_i$ is not equal to the difference $r_i - r_{i-1}$: the motion is not uniform, but the velocity changes uniformly with time. In fact, we verify that the right expressions for the velocity and the location are reproduced. In hindsight, this is obvious. The contribution $\frac{1}{2}a\Delta^2$ is the displacement due to acceleration a during time lapse Δ.

We can also achieve a consistent result using our order-2 formula, Expression 3.12, as follows:

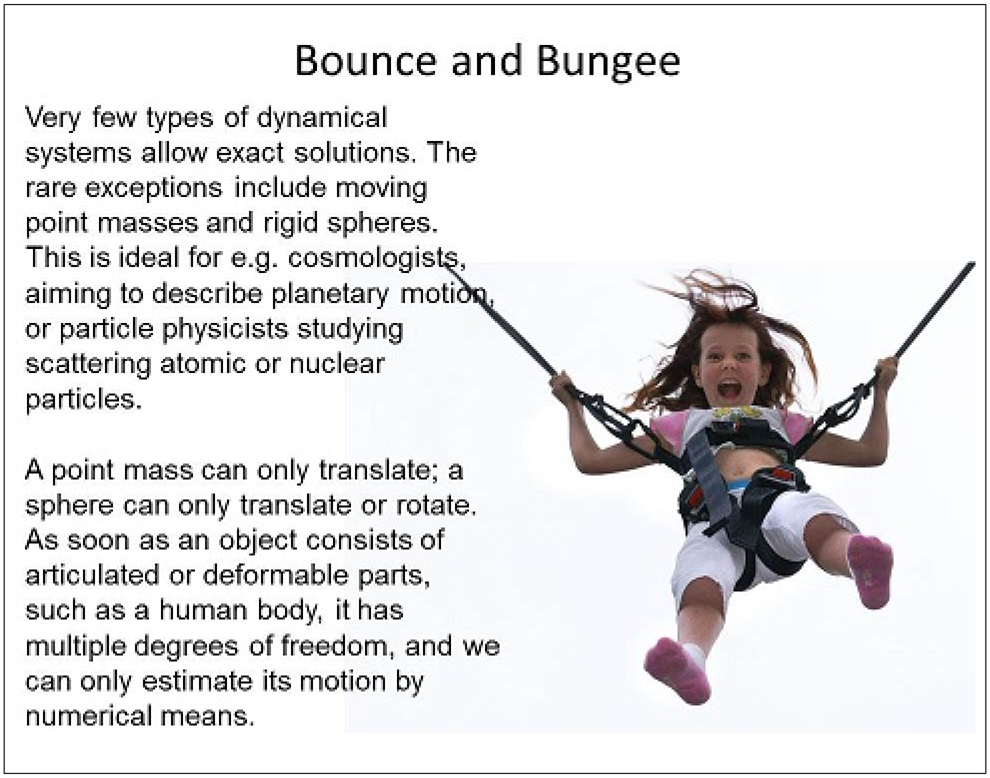
$$r_{i+1} = 2r_i - r_{i-1} + a\Delta^2. \quad (3.22)$$

Apparently, the order-2 recursive model of Expression 3.22 describes a point masses subject to acceleration a. For a=0 we get the result conform hight school physics for a uniform motion. For a=constant, we get the motion of a uniformly accelerated point mass.

We postulate $^{\triangleright 72}$ that, in a recursive model, $r_{i+1} = 2r_i - r_{i-1} + a_i\Delta^2$, a_i is the acceleration at time step i. So whenever we should simulate the motion of a point mass, subject to a force, constant or varying, we try the recursive function Expression 3.22.

[21] The image of an old Gdansk candy shop is taken from http://commons.wikimedia.org/wiki/File:Gdansk_Jana_z_Kolna_napis.jpg?uselang=nl

3.3.4 Totally Ordered Time; Equal, Infinitesimal Intervals: Continuous Time



The behavior for physical, financial, and many other types of systems can be modeled with recursively defined models[22]. Running such models amounts to repeated evaluation of the recursive functions to unroll the process. Sometimes this gives an exact result. This is possible if the modeled system only has transitions at discrete moments, such as financial systems. Also for physical systems recursive models can give an exact result, that is: a result that is independent of the choice of Δ and matches the results from physics.

We have also seen cases, however, where the result does depend on Δ, whereas the modeled system does not contain discrete transitions. In these cases, such as the rotating dumbbell and the mass-spring system, detailed in End Notes 73 and 74, the result may converge if we take $\Delta \to 0$.

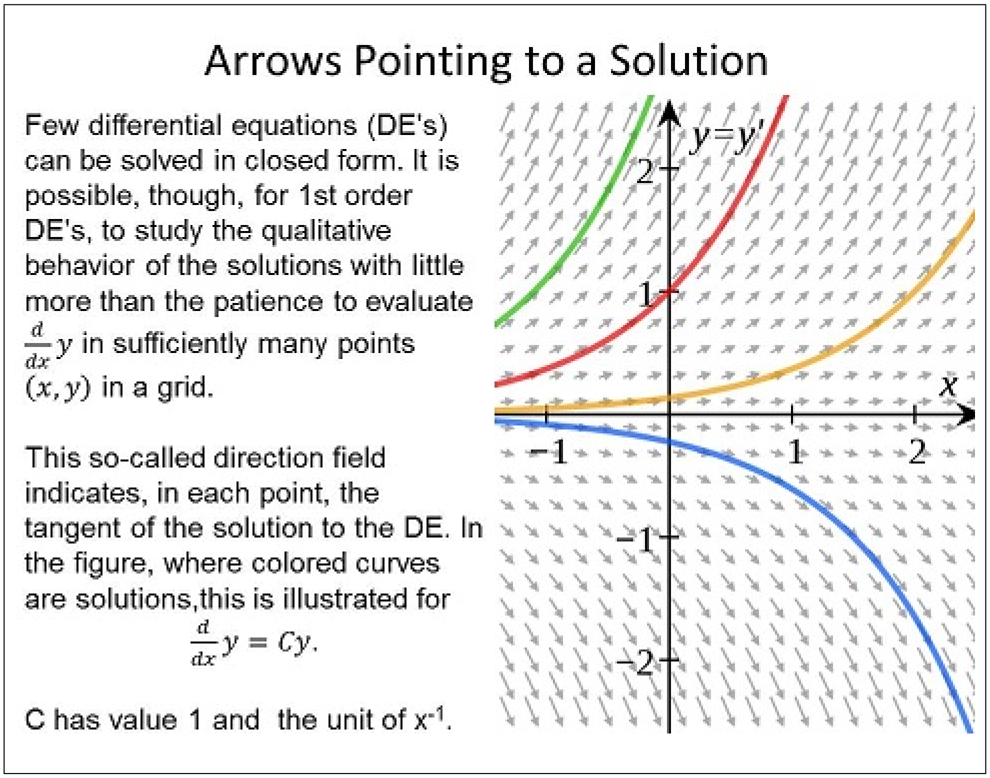
This suggests that unrolling the recursive function is the numerical solution of a DIFFERENTIAL EQUATION. It turns out that numerical schemes using recursive functions can be devised ▷73 to help solve differential equations, also for more advanced dynamical systems.

To model dynamic systems, we have various routes at our disposal ▷74 . We can try to describe their behavior by a differential equation and attempt to solve this by analytic means, or we can try to form a recursive function and unroll the recursion to simulate the system. A third, perhaps even more common approach is, to derive the differential equation and find an approximate solution by numeric means.

Both symbolic and numeric methods have their advantages and disadvantages; see also Table 1.2. In case we are only interested in numerical estimates, and perhaps an indication of their accuracy, numerical methods are the best choice - although maybe at the expense of substantial computational effort. Numerical schemes are often simple to derive, and they are robust against small modifications of the precise form of the model. To the contrary, symbolic methods give deeper insight in the structure of the studied system, they typically don't require extensive computer processing, but they are much more selective as to their application: the smallest modification of a problem may render an analytic technique useless.

[22] The photograph of a bungee girl is taken from http://www.rgbstock.nl/download/BlueGum/mflfx8M.jpg

3.4 Mathematical Tools: Numerical Methods for Differential Equations



In Section 3.3.2, we have seen that recursive functions can be used to evaluate or unroll a behavior. If infinitesimal intervals (i.e. taking the limit for Δ to 0) are chosen, recursive functions become differential equations (see Endnote 74). In the basic course Calculus differential equations were discussed. Some methods to solve differential equations were given. However, many differential equations cannot be solved exactly. In this section Euler's method to approximate the solution of a differential equation will be examined. To introduce this we will first discuss direction fields.

3.4.1 Direction Fields

In Figure 'Arrows Pointing to a Solution' a direction field is depicted[23]. A direction field can be found in the following way. Consider the differential equation:

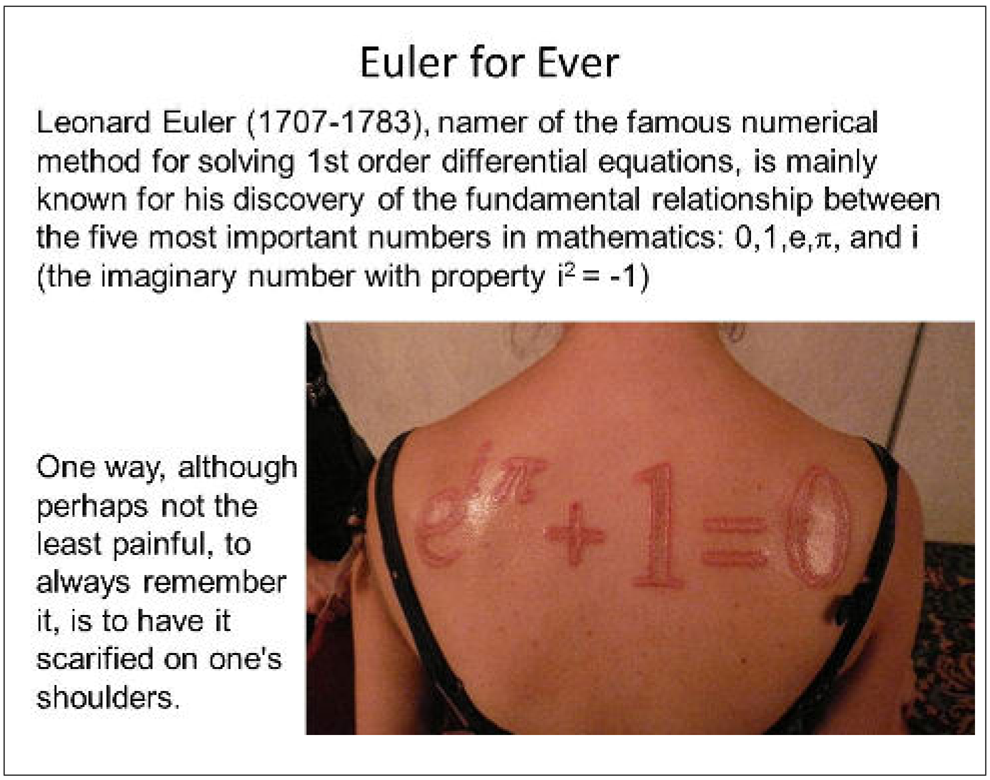
$$y'(t) = Cy(t), \tag{3.23}$$

which describes, among other things, the growth of bacteria. C is a constant with unit=1/unit of t. The exact solution of this differential equation is given by $y(t) = Ae^{Ct}$, where the constant A can be found if an initial value is given (for example $y(0) = 2$). In a direction field an arrow starting at a point $(t_0, y(t_0))$ indicates the instantaneous rate of change of a specific solution for a given value t_0. This instantaneous rate of change is equal to the slope of the tangent line to the solution curve at the point $(t_0, y(t_0))$. Therefore it equals $y'(t_0) = ACe^{Ct_0}$. So, for a given value

[23]The image of a direction field was taken from http://commons.wikimedia.org/wiki/Category:Differential_equations#mediaviewer/File:DGL_y-eq-dy.svg

of t_0 several arrows can be given by choosing different values of A. The arrows each are tangent to one of the exact solutions. Go to this link to interactively play with direction fields for Expression 3.23. By clicking anywhere within the direction field image, one of the solutions is seen to develop.

One does not need to know the exact solution of the differential equation to draw a direction field. As follows. Consider a point (t_0, y_0) and suppose that this point is part of a SOLUTION CURVE. So, for some initial value, there exists a solution $y(t)$ of the differential equation with $y_0 = y(t_0)$. We know, using the differential equation, that $y'(t_0) = Cy(t_0)$ (see Expression 3.23) at $t = t_0$. Therefore, the slope of the tangent line to the solution curve at (t_0, y_0) is equal to $Cy(t_0)$. So, e.g. for $C = 1$, if we choose for example the point $(2, 4)$, we know that the instantaneous rate of change is equal to 4 and we can draw an arrow in the point $(2, 4)$.



The direction field gives an impression of the behavior of the solution curves. The solutions, however, cannot be derived from the direction field, neither numerically nor in closed form. This has to do with the fact that an arrow gives the direction of change for a given point (t_0, y_0), but we do not know which other points in the plane belong to the same solution curve. But we do know that the line through the arrow also represents a linear approximation of the function. So if we choose points on the line close to the point (t_0, y_0), these points might be close to points that belong to the actual solution. This idea is used in Euler's method.

3.4.2 Euler's method

Consider[24] the following differential equation with given initial value

$$y'(t) = f(t, y), \qquad \text{with} \qquad y(t_0) = y_0. \tag{3.24}$$

[24] The photograph of a scarified version of Euler's relation is taken from http://upload.wikimedia.org/wikipedia/commons/3/36/Euler%27s_identity_scarification%2C_3PiCon%2C_Springfield%2C_MA.jpg?uselang=nl

The equation of the tangent line to the solution curve at the point (t_0, y_0) is given by

$$y = y_0 + y'(t_0)(t - t_0). \tag{3.25}$$

We can use this to find an approximate value y_1 of the solution at $t = t_1$. This approximate value is the y-coordinate of the point on the tangent line for $t = t_1$. So,

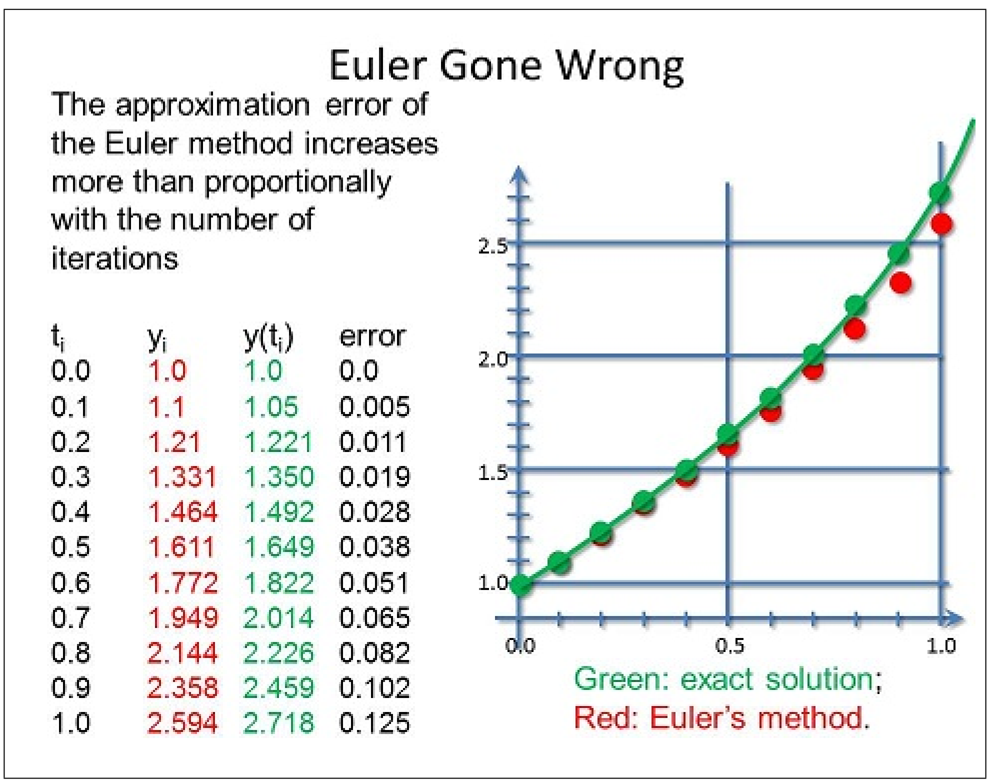
$$y(t_1) \approx y_1 = y_0 + y'(t_0)(t_1 - t_0). \tag{3.26}$$

This approximation is in general better if the value of t_1 is closer to the value of t_0. The value y_1 can be used to find an approximation of the value $y(t_2)$ at $t = t_2$

$$y(t_2) \approx y_2 = y_1 + y'(t_1)(t_2 - t_1). \tag{3.27}$$

This process can be repeated. The values $t_1 - t_0$ and $t_2 - t_1$ in Expressions 3.26 and 3.27 are called *step sizes*. Typically, the same step size is chosen throughout; it is denoted by h. The value of the derivative can be found with the differential equation given in Expression 3.24. Therefore

$$y(t_1) \approx y_1 = y_0 + h \cdot f(t_0, y_0), \qquad y(t_2) \approx y_2 = y_1 + h \cdot f(t_1, y_1). \tag{3.28}$$



Continuing in this way, Euler's method finds the sequence of approximate values

$$y_{i+1} = y_i + h \cdot f(t_i, y_i), \tag{3.29}$$

approximating $y(t_{i+1})$, where $i = 0, 1, 2 \ldots$.

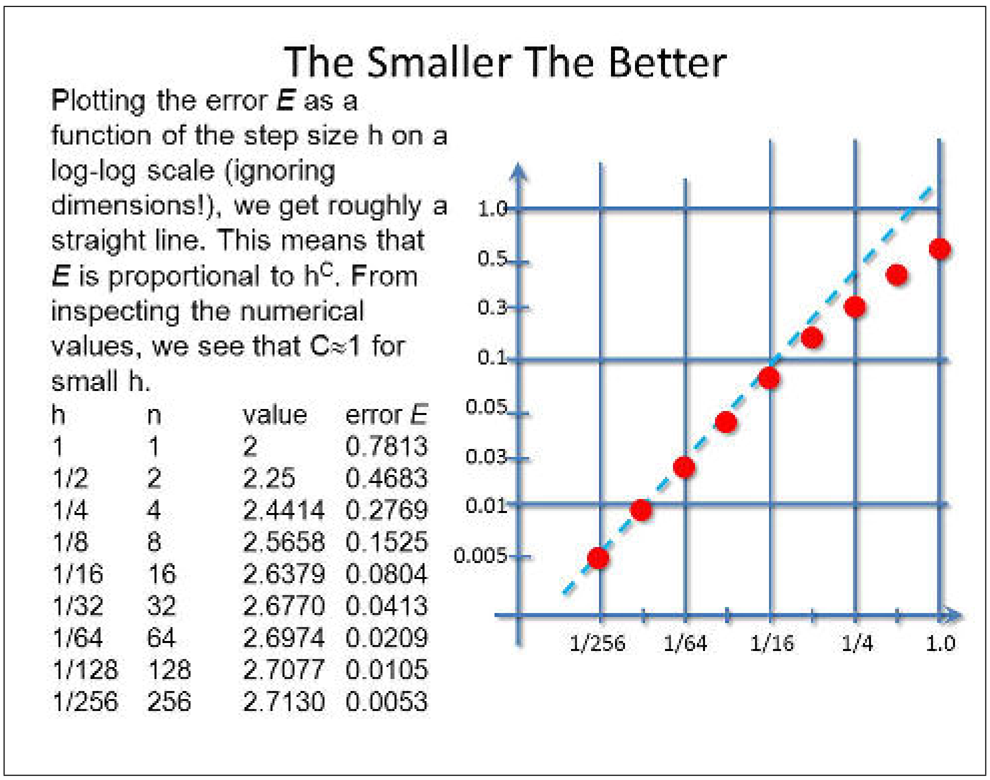
In Section 'Time Lapses with Equal Length; Sampling' (3.3.3) we have seen how the behavior of a dynamical system can be obtained by unrolling a recursive function as from Expression 3.3, $Q_i = F(Q_{i-1}, P_{i-1})$. Notice that Euler's method has the exact form of unrolling a recursive function. The difference between the approach in earlier sections and Euler's method is, that in earlier sections we did not attempt to write the dynamical model in the form of a differential equation: rather, we wrote the (glass box) mechanisms directly in the form of recursive functions.

Both approaches have their strengths and weaknesses:

• *skipping the formulation of a differential equation* often gives a straightforward derivation of the recursive functions. The solution (=finding the dynamical behavior of the modelled system) does not require any analytical skills or knowledge of numerical methods. The disadvantage is, that the step size needed for sufficiently accurate approximations may have to be to be extremely small, leading to very large numbers of iterations.

• *formulating the dynamical system in the form of a differential equation* may require more advanced physical, mechanical or chemical reasoning. It may also, however, in some cases, lead to differential equations that can be solved in closed form by analytical means. And if this is not possible, there usually exist numerical methods that are more efficient than Euler's method.

Most cases of unrolling recursive functions, as discussed in this chapter, are in fact based on Euler's algorithm ▷75 .



In the following we use Euler's method to find an approximate solution of the differential equation given in Expression 3.23, where we take C=1, with initial value $y(0) = 1$. Since in this case the exact solution is known, one can compare the approximate solution with the exact solution.

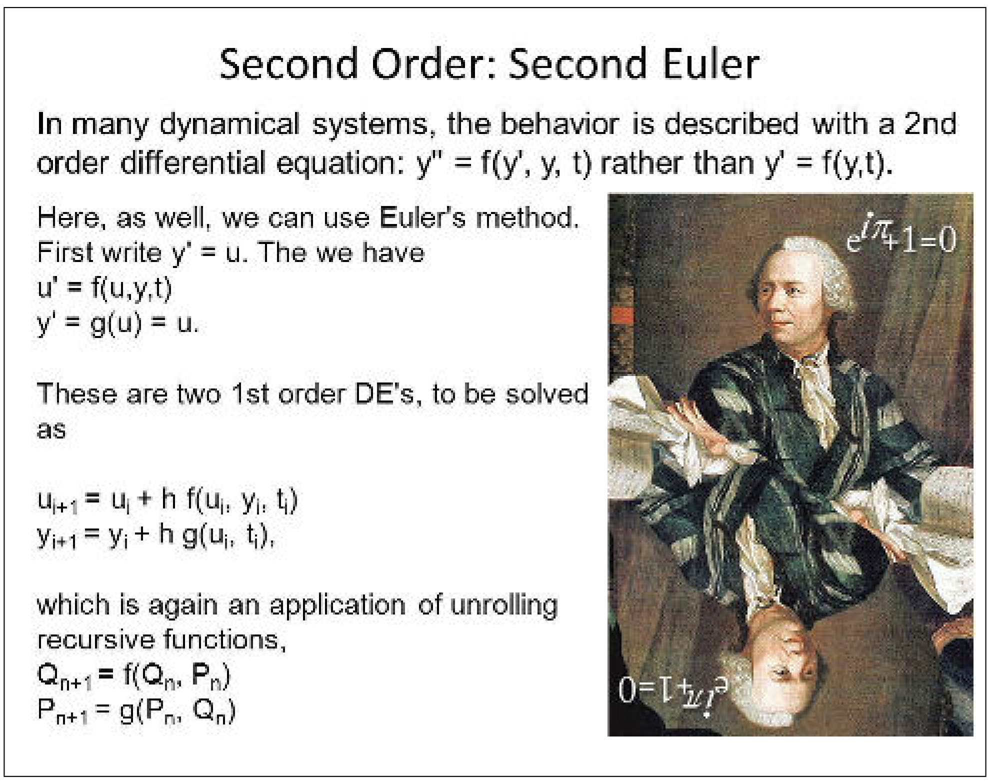
With step size $h = 0.1$ one finds

$$\begin{aligned} y(t_1) &\approx y_1 = y_0 + h \cdot f(t_0, y_0) = 1 + 0.1 \cdot 1 = 1.1, \\ y(t_2) &\approx y_2 = y_1 + h \cdot f(t_1, y_1) = 1.1 + 0.1 \cdot 1.1 = 1.21, \\ y(t_3) &\approx y_3 = y_2 + h \cdot f(t_2, y_2) = 1.21 + 0.1 \cdot 1.21 = 1.331. \end{aligned}$$

In Figure 'Euler Gone Wrong', the first ten values are given. We see that the error increases with the value of the argument t. In Figure 'The Smaller The Better', the result is shown for other step sizes. The error increases slower if the step size is smaller. In fact, the error in estimating

$y(1)$ is seen to be more or less proportional to the step size.

One could (and should!) of course ask 'which step size is small enough'. This first depends on the desired accuracy of the approximated function value; this will be discussed further in Chapter **??**. It also depends, however, on the rate of change of the solution we try to approximate. If the solution varies slowly, step sizes may be larger compared to the case where a solution shows rapid changes. In general, the step size should be small compared to the CHARACTERISTIC TIME. Indeed, many dynamical processes can be associated to a characteristic amount of time. For periodic processes, this is the period (e.g., for a beating heart, the characteristic time is the average duration of one heart beat). For (exponentially) increasing or decreasing behavior, it is e.g. the time of doubling or halving. For the behavior to be reproduced at least qualitatively, there should be several time steps within the characteristic time of the modeled process. So, for a differential equation with periodic solutions, the time step should be (much) smaller than the period.



Notice that this relates to dimensional analysis: the unit-less ratio $\frac{h}{T_C}$ of the step size h and the characteristic time, say T_C should be (much) smaller than 1. The number n of iterations needed to calculate an approximation to the solution for a given time t should therefore be inversely proportional to T_C. Moreover, for the Euler method we see that the error, for a given h, increases more than proportional to the duration of time for which we want to find the solution. For the purpose *prediction* we could call this the PREDICTION TIME, T_P. The number of iterations is therefore at least proportional to T_P. Combining the latter two results, the number of required iterations is at least proportional to $\frac{T_P}{T_C}$.

Finally, in Figure 'Second Order: Second Euler'[25], we demonstrate that with little effort, Euler's method can also be used for differential equations including 2nd order derivatives, such as the equations occurring in physics and electronics to describe the motion of objects under the influence of forces ($F = ma$), or the behavior of electric currents in networks of resistors, capacitors and coils.

[25]The original portrait of Euler was taken from http://upload.wikimedia.org/wikipedia/commons/6/60/Leonhard_Euler_2.jpg?uselang=nl

3.4.3 Equilibrium Solutions and Stability

A solution $y(t)$ of the differential equation given in Expression 3.24 is called an equilibrium solution if it is a constant function. So, $y'(t) = 0$ and from Expression 3.24 it follows that $f(t, y(t)) = 0$ for all t.

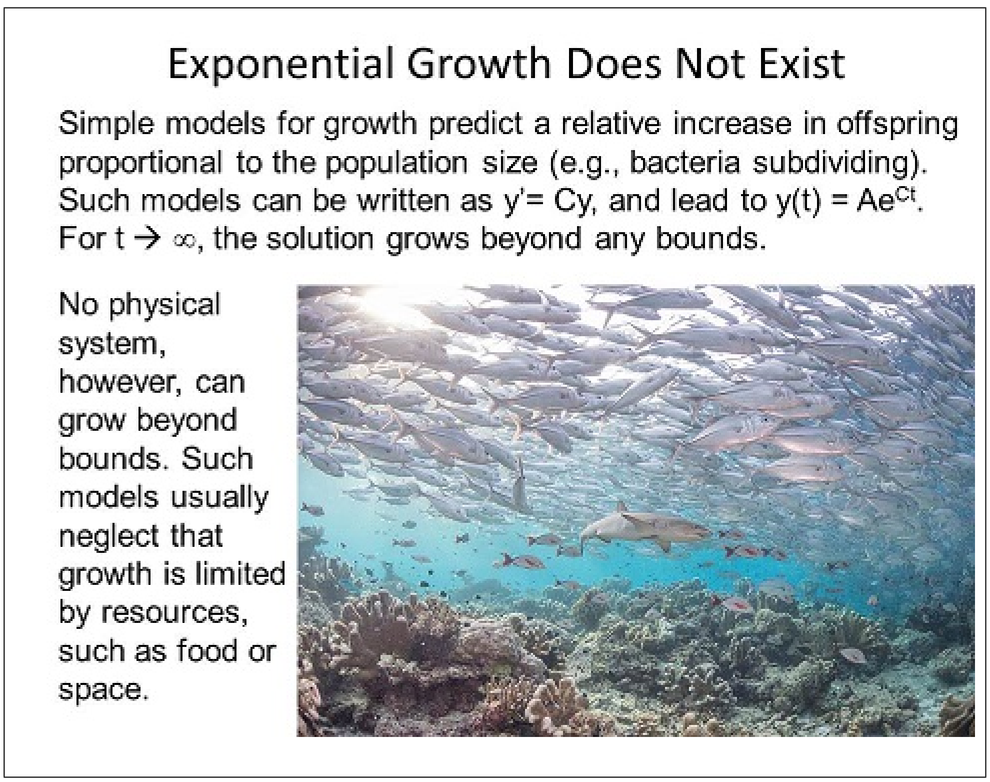
Consider the differential equation

$$y'(t) = k \cdot y(M - y), \tag{3.30}$$

which is a model for so called LOGISTIC GROWTH[26] where M is the maximum population determined by resources (with unit the unit of y) and k is the growth rate, with the unit of y per unit of time. For equilibrium solutions we must have $y'(t) = 0$ and so $y(t)(M - y(t)) = 0$. This gives $y(t) = 0$ or $y(t) = M$ for all t. Go to this link to experiment with line element fields for $y'(t) = k \cdot y(M - y)$. Adjust k and M at will; click somewhere in the image to follow a solution, starting in the clicked location.

One can see that for the line $y = M$ all arrows near the line point towards it, whereas the arrows at the line $y = 0$ diverge. Therefore the solution $y(t) = M$ is a so-called STABLE SOLUTION. An equilibrium solution is *stable* if solutions close to the equilibrium approach the equilibrium solution for $t \to \infty$. An equilibrium is *unstable* is solutions close to the equilibrium solution tend to get further away from the equilibrium if $t \to \infty$. $y = 0$ is therefore an unstable equilibrium.

Mathematical definitions for these concepts are not given here. In practice, the (un)stability can be checked by use of the direction field.



[26]The photograph of a school of fish was taken from `http://commons.wikimedia.org/wiki/File:Reef_shark_beneath_a_school_of_jack_fish.jpg?uselang=nl`

3.5 Summary

- A *state* is a snapshot of a conceptual model at some time point;
- The *state space* is the collection of all states of a modeled system;
- Change comes in the form of *transitions* between states; a *state chart* is a graph where nodes are states and arrows are transitions;
- A *behavior* is a path through state space; a process is the set of all behaviors. The size of state spaces is huge, causing practical problems for state space models; this is called *state space explosion*. Two methods to mitigate state space explosion:
  - *symmetry*: some parts of state space are identical and therefore redundant;
  - *projection*: distinguish *exposed* and *hidden* properties or value sets and if possible limit the state space to exposed properties only;
- Multiple flavors of time:
  - *partially ordered time*, for instance for specification and verification;
  - *totally ordered time*, for instance for prediction, steering and control;
  - a *recursive* function of the form $Q_{i+1} = F(Q_i, Q_{i-1}, Q_{i-2}, ..., P_i, P_{i-1}, P_{i-2}, ...)$ is used to evaluate or *unroll* a behavior;
    * *arbitrary, perhaps unequal intervals*: often no methods for closed form evaluation; unrolling is the only approach;
    * *equal intervals*: the possibility for closed form evaluation (example: periodic financial transactions); *sampling*;
    * *equal, small intervals*: approximation, sampling error (examples: moving point mass, rotating dumbbell, mass-spring system, dissipation);
    * *infinitesimal intervals*: *continuous* time, differential equations (examples: motion of a point mass with or without force); contrast between numerical and symbolic approach
- Numerical methods for differential equations:
  - construction and interpretation of a direction field;
  - Euler's method to find approximate solutions of first-order differential equations;
  - equilibrium solutions.

3.6 Learning goals

3.6.1 Knowledge

You should know the meaning of the terms state, state diagram, transition, process, behavior, event, state space explosion, deadlock, projection, hidden and exposed properties; the various types of time models (partially ordered time, total ordered time, with or without equal time intervals); you should know what recursive functions are, and how to use them, and what sampling is. You should know the main ideas behind numerical and symbolic solution of differential equations, their use, application, advantages and disadvantages.

Adams: §18.3 without the 'improved Euler method' and without the 'fourth-order Runge-Kutta method'.

Smith & Minton: §7.3.

3.6.2 Skills

This chapter contains numerous examples. For each of the examples, you should be able to invent at least two very different other ones. You should be able to set up a state chart for an arbitrary discrete dynamic system (with a limited amount of states). You should be able to distinguish permitted and forbidden transitions; you should be able to choose among various alternative choices for state properties. You should be able to calculate the number of states in a state chart. You should be able to choose hidden and exposed properties, and reason about the consequences of your choice. You should be able to recognize and, for simple dynamic systems, devise a recursive function. Also for simple dynamical systems, you should be able to unroll a recursive function (using ACCEL or an other compute environment); you should be able to choose a stepsize and understand its consequence in terms of accuracy and computation time. Even if you may not be able to actually solve a differential equation by analytic means, you should be able to make a substantiated choice between numerical and symbolic methods for modeling a dynamic system, given the purpose you try to achieve. Notice: the mathematical treatment of the rotating dumbbell and the mass-spring system, elaborated in End Notes 73 and 74, are not part of the mandatory material.

For first-order differential equations, you should understand the meaning of a direction field and be able to interpret a given direction field. You should be able to plot a direction field in simple cases. You should understand Euler's method to approximate solutions of first-order differential equations and know its limitations. You should be able to find equilibrium solutions.

3.6.3 Attitude

When confronted with a problem that involves change over time, you should be inclined to think about it in terms of states and state transitions. For discrete systems, you should be inclined to think in terms of state charts; you should have the tendency to try and find representations that lead to transparent and possibly small state spaces. For systems with totally ordered time, you should be inclined to search for recursive functions; when asked to implement a simulation of a dynamical system, you should either be tempted to seek help to find a closed-form solution, or try to implement an algorithm to unroll the appropriate recursive function, both depending on the purpose of your model. When using a numerical method, you should be inclined to think about accuracy in terms of step size, and you should do experiments to try and estimate the accuracy of the numerical outcomes.

3.7 Questions

1. In the first paragraph of Section 3.2, a number of occurrences in relation to lights and radios are mentioned. Which of these are states, which are events?

2. For each process mentioned in the first paragraph of Section 3.2, draw a little state chart.
3. We say 'Applying butter is to be synchronized with the other two stages of the process'. Say this in your own words.
4. What is the use of a state chart?
5. Draw a state chart for making tea that takes into account that at any time the telephone might ring.
6. What do we mean by 'binding'?
7. How long does a state transition take?
8. A system has N_p properties, each of type `Boolean`. How many behaviors of b steps are possible?
9. What is the difference between a behavior and a process?
10. Describe one shot in a game of billiard as a state chart.
11. Explain the notion of 'symmetry' in your own terms.
12. What is the meaning of 'reachable' in state spaces?
13. Hiding properties can cause a system to behave seemingly random. Explain.
14. What is lifelock?
15. What is the difference between an event and a transition?
16. Why is it allowed to assume that no two events can occur at the same time?
17. What is an internal transition?
18. In each of the three examples of internal transitions in Section 3.3.1, think of one or more hidden properties that should be made exposed so that the transitions are no longer internal.
19. For the three examples of partial ordering in Section 3.3.1, draw part of a state chart that illustrates the partial ordering.
20. In the case of totally ordered time, we enumerate both states and transitions. Why is this allowed?
21. In a dynamic model with totally-ordered time, t_i, t_j and t_k are times where transitions occur. From $\Delta(t_i, t_j) + \Delta(t_j, t_k) = \Delta(t_i, t_k)$ it follows that transitions take no time. Explain.
22. Explain in your own words: what is recursion?
23. (*) Causality forbids that the cause of some occurrence takes place after the occurrence. Is causality a sufficient condition for the existence of a recursive function to simulate a dynamic process?
24. Explain why a recurrent function of the form $Q_{i+1} = F(Q_i, P_{i+1})$ is forbidden.

25. Explain the meaning of 'order' in 'the order of a dynamical system'.

26. (*) You may know what the 'order of a differential equation' means. If so, explain the relation between this meaning of 'order' and the 'order of a dynamical system described by a recurrent function'.

27. We claim that for a system, described by a recurrent function of the form $Q_i = F(P_{i-1})$, everything is static. Explain.

28. Give an example of a system with order > 2. (Hint: don't think of a physical system).

29. In Section 3.3.3 we say that a fraction $s_0\Delta_0$ is spent in time interval Δ_0. Explain the product $s_0\Delta_0$.

30. In the financial example in Section 3.3.3, we calculate how much we are allowed to spend for a given income. We could also ask the question: how much do we need to earn for a given spending rate. Re-do the example to answer this latter question.

31. We given an example of recursion to calculate $N!$. How many multiplications does this recipe take? How can you do it quicker?

32. In the closed-form calculation of the financial example, we spend a fraction $s\Delta$. Redo the example where we *save* a fraction $s\Delta$.

33. Schemes for unrolling recursive functions may introduce an error. The example for a moving point mass with constant velocity we derived in the text had order 1. What does this mean?

34. Schemes for unrolling recursive functions may introduce an error. The example for a moving point mass with constant velocity we derived in the text had order 1. For what reason is a scheme for unrolling recursive functions that has order 2 better than a scheme with order 1?

35. Verify that the scheme from Expression 3.22 for unrolling recursive functions for the accelerated point mass introduces no error.

36. Verify Expression 3.19.

37. In the example of the rotating dumbbell, the solution Expression B.48 contains a factor $\frac{1}{2}$. You may remember the formula for centrifugal force, $F = m\omega^2 r$, which contains no such factor. Explain why both formulas are consistent.

38. In problems such as the rotating dumbbell, there is a maximum time step Δ. Relate this maximum time step to the rotation frequency.

39. We propose, in Expression B.55, to represent damping by $\epsilon(r_i - r_{i-1})$. Show that this indeed gives a reduction of $(1-\epsilon)^2$ per time step of the kinetic energy.

3.8 Exercises

1. In at least two different ways, describe one shot in a game of billiard as a state chart (for both, formulate a purpose - the purposes don't need to be very plausible).

2. In the text, we give three examples where symmetry helps reducing the size of state space. Which three? Give a fourth example.

3. Consider the four color ballpoint described in Appendix **??**.

   (a) The number of permitted states of the four color ballpoint is 5/32 of the number of possible states. Explain.

   (b) Change the conceptual model of the four color ballpoint so that all possible states are permitted.

4. The externally visible behavior of a baby can be characterized by two properties: `sound:[cry,silent]` and `consciousness:[sleep,awake]`

   (a) Make a state chart with the states and transitions among these properties.

   (b) For each transition, indicate whether it can be causally understood or not.

   (c) Babies often cry because their stomach is empty. Add a (hidden) property `stomach:[full, empty]`, and extend the state chart accordingly.

   (d) Give an interpretation of each transition: is it caused by internal events or external events?

   (e) Think of a purpose for which the latter state chart would be a useful model.

   (f) Think of an extension of the model such that the purpose of Question 4e is fulfilled better (you don't need to give the full state chart for your extended model).

5. Consider a coin-operated coffee machine

   (a) Propose a purpose for a dynamic model.

   (b) What properties of the coffee machine need to be taken into account, and what are their value sets?

   (c) Set up a state chart for the coffee machine.

   (d) Propose a property to be hidden. Which part of the behavior then becomes 'random' (=unpredictable)?

   (e) Propose an extension of the model such that your purpose is fulfilled better (you don't need to give the full state chart for your extended model).

6. (a) Propose a system (other than asock, a baby, a four color ballpoint and a coffee machine), and a purpose that would merit a dynamical model.

   (b) Propose a small, but meaningful set of properties for your system such that a statechart model would help to realize the purpose.

   (c) Propose the smallest, but meaningful sets of values for these properties.

   (d) Give a state chart for the dynamical behavior of the model.

(e) Think of an extension that would better serve the purpose (you don't need to give the full state chart for your extended model).

7. Give a state chart of the game of tick tack toe; make use of symmetry to reduce the number of states.

8. In Section 3.3.2 we introduce a simple model for a cycling tour. This model is naive. Why? Suggest at least two improvements. Elaborate these improvements in the form of recursive functions.

9. Give an example of a dynamical system with an order larger than 2.

10. The Watt regulator is a system to control the speed of a steam engine. It consists of two weights, connected on a vertical shaft that rotates with a speed equal to the speed of the flywheel. Call the rotation speed ω. The weights are mounted on hinges that allow them to move outward due to centrifugal force, and at the same time go up. The height of the weights is h, and h depends on ω. The height of the weights is used to control a valve, which regulates the steam pressure P: when the weights go up, the steam pressure is lowered and vice versa. So P depends on h. Finally, the rotation speed ω depends on P and on the external load L: ω increases with P and decreases with L.

    (a) Give the simplest possible functions that express how h depends on ω, how ω depends on P and L, and how P depends on h.

    (b) Show that these three functions are not in the form of a set of recursive functions of Expression 3.2.

    (c) Give a function that expresses a *change* in ω in terms of a *change* in P and L, and similar for h and P.

    (d) Show that the latter functions are in the form of Expression 3.2.

    (e) (*) Show that the Watt regulator stabilizes the rotation speed of a steam engine - that is: when the rotation speed changes due to a change in L, the rotation speed will soon get back to the original speed.

    (f) Discuss limitations in the functioning of the Watt regulator according to the model.

11. In Endnote 73 we give a model for a mass spring system using discrete time steps. Derive a similar model for a pendulum.

    - First, assume that the deviation angle ϕ is small enough so that we may approximate $\sin(\phi)$ by ϕ.
    - Show that the approximated solution is independent of the maximum deviation.
    - Show that, for sufficiently small time step Δ, the approximated solution converges to the closed form solution.
    - Next, drop the assumption of small maximal deviation. Again, give the discretely approximated solution.
    - (*) In the example of an oscillating mass-spring system, we saw that there is a condition relating the time lapse Δ to the frequency of resonance, $\sqrt{\frac{C}{m}}$ of the mass spring system: the approximation only works if Δ is sufficiently small compared to $\sqrt{\frac{m}{C}}$. Can you find a similar condition for the case of the pendulum?

12. There are two armies; their numbers of soldiers initially are S_1 and S_2. When they engage in a battle, it takes e_1 soldiers of army 1 to kill one soldier of army 2, whereas it takes e_2 soldiers of army 2 to kill one soldier of army 1. Devise a dynamical model, in the form of recursive formulas for S_1 and S_2 that depend on time, such that we can predict the outcome of the battle.

13. In a biotope, there lives a population of fox and a population of rabbit. The sizes of the two populations are F and R; both depend on time, and their initial values are F_0 and R_0. We want to predict if the two populations can continue to live together in the same biotope.

    (a) Which factors influence next years population sizes?

    (b) How do the simplest mathematical relations look like to account for these factors?

    (c) Devise a discrete dynamical model, using recursive functions, to describe the evolution of F and R over time.

Exercises concerning Mathematical Tools (Section 3.4)

Adams: §18.3: to be decided.

Smith and Minton: §7.3: to be decided.

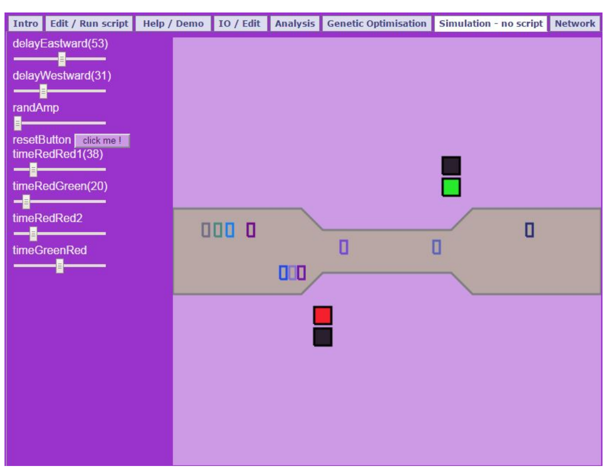
Appendix A

Average Waiting Time at the Traffic Lights

'Patience is a virtue'

A.1 Traffic Lights on a road with one Lane

In Section 1.2.2 we mentioned the traffic lights problem. One lane of a road is blocked due to repair work. Traffic lights are needed to regulate the traffic.



Here the length of the red and green period of traffic lights have to be set. We have the following pattern for the traffic lights.

direction I	R_0 G_1 R_0 R_1
direction II	R_0 R_2 R_0 G_2

As purpose is chosen to minimize the average waiting time per car. We choose a model for the traffic flow to estimate the average waiting time for a car. Suppose that f_i is the flow of traffic in direction i in number of cars per minute $(i = 1, 2)$ and f_0 is the number of cars per minute that can pass the part of the road with one lane. Furthermore, let R_0 is the time both traffic lights are red simultaneously and R_i is the time the traffic light in direction i is red and the other traffic light is green $(i = 1, 2;\ R_1 = G_2; R_2 = G_1)$.

For the model we assume that the cars arrive in a deterministic pattern with a constant inter arrival time (see also the discussion in Section 1.2.2). Furthermore we assume that the variables above have such values that there is no queue anymore at the moment that the traffic light turns red again.

Let N_1 and N_2 be the numbers of cars that have to wait for the traffic light in direction 1 and 2. We compute the numberN_1 in two ways.

Dividing G_1 in two parts. The total numbers of cars that arrive during one cycle equals $(2R_0 + R_1 + G_1)f_1$. These cars must pass the traffic lights during the time the traffic light is green G_1. The cars that have to wait need a time N_1/f_0 to pass the traffic lights. The number of cars that do not have to wait equals $(2R_0 + R_1 + G_1)f_1 - N_1$ and the time that it takes to pass the traffic light is for these cars $[(2R_0 + R_1 + G_1)f_1 - N_1]/f_1$. Therefore we have

$$\frac{N_1}{f_0} + \frac{(2R_0 + R_1 + G_1)f_1 - N_1}{f_1} = G_1. \tag{A.1}$$

Using the fact that $G_1 = R_2$ we find

$$N_1 = (2R_0 + R_1)\frac{f_1 f_0}{f_0 - f_1}. \tag{A.2}$$

Using a geometrical series. The number of cars that arrive in direction 1 during a period $2R_0+R_1$ equals $(2R_0+R_1)f_1$. So, when the traffic light turns green again, the number of cars in the queue equals $(2R_0+R_1)f_1$. However, the time it takes to dissolve this queue is $(2R_0+R_1)f_1/f_0$. During this period the number of cars that arrive is $(2R_0 + R_1)f_1^2/f_0$, and these cars have to wait also. Again, this takes a time of $(2R_0 + R_1)f_1^2/f_0^2$. We can continue this argument. The total number of cars that have to wait can be seen as the sum of a geometrical series with ratio f_1/f_0 and we find

$$N_1 = (2R_0 + R_1)f_1 + (2R_0 + R_1)\frac{f_1^2}{f_0} + (2R_0 + R_1)\frac{f_1^3}{f_0^2} + \cdots \tag{A.3}$$

The sum of this geometrical series is equal to the value in Expression A.2.

Now we compute the total waiting time. The first car that arrives in direction 1 at the moment that the traffic light turns red again has to wait $2R_0 + R_1$ minutes. The last car that arrives in the queue does not have to wait. So the average waiting time for these cars is $(2R_0 + R_1)/2$ and the total waiting time in direction 1 is $N_1 \cdot (2R_0 + R_1)/2$. The total waiting time for the cars in both directions is $N_1 \cdot \frac{1}{2}(2R_0 + R_1) + N_2 \cdot \frac{1}{2}(2R_0 + R_2)$. So we have for the total waiting time T

$$T = \tfrac{1}{2}(2R_0 + R_1)^2 \frac{f_0 f_1}{f_0 - f_1} + \tfrac{1}{2}(2R_0 + R_1)^2 \frac{f_0 f_2}{f_0 - f_2}. \tag{A.4}$$

The total number of cars that pass during a cycle equals $(2R_0 + R_1 + R_2)(f_1 + f_2)$.
Therefore the average waiting time per car equals

$$F = \frac{f_0}{2(f_1 + f_2)} \cdot \frac{\frac{f_1}{f_0 - f_1}(2R_0 + R_1)^2 + \frac{f_2}{f_0 - f_2}(2R_0 + R_2)^2}{2R_0 + R_1 + R_2}. \tag{A.5}$$

The no queue condition

For the traffic lights model we have the constraint that any traffic light should be green long enough to ensure that all the cars can pass the road. The number of cars that arrive in direction 1 during one cycle equals $(2R_0 + R_1 + R_2)f_1$. This number of cars must pass during the time the traffic light is green. This time is $G_1 = R_2$. Therefore we have

$$(2R_0 + R_1 + R_2)f_1 \leq R_2 f_0,$$

and

$$(2R_0 + R_1 + R_2)f_2 \leq R_1 f_0.$$

Appendix B

Formal aspects of Conceptual Modeling

'To assure that your readers won't be confused, write as if you program a computer'

Chapter 2 discusses how to assemble a conceptual model. Often we will have to translate (parts of) the conceptual model into a form, suitable for processing in a computer. This asks for additional precision. We have to make unequivocally clear which properties belong to which concepts, and which relations apply between them.

Therefore we need to pay attention to the notation and further formalities of concepts, properties and relations.

B.1 Relations and Arities

An important notion regarding relations is their ARITY. The arity of a relation defines the number of concepts involved in each TERM of the relation. If a relation takes two arguments, it is called a BINARY RELATION. Many binary relations are 1-1, for instance the traditional Western marriage with 1 male and 1 female partner. A polygamous marriage (1 male, multiple female partners) is 1-n, whereas the rare polyandric marriage (1 female, multiple male partners) is n-1. A 1960-ies hippy commune, with multiple male and female partners is an example of an n-m relation. In Section 2.5, the arity was indicated by the numbers in parentheses after the concept names. In figure 2.1 the arities were written next to the arrows, representing the relations.

Binary relations require two terms; it is possible for a relation to have three or more, though. An example is the relation `see` in figure 2.1: seeing is only possible with a viewer (the concept `driver`), something to look at (the concept `road`), and one or more light sources (notice the 'n' near the beginning of the arc to the concept `lantern`). So the arity of the relation `see` is 1-1-n for `driver`, `road`, `lantern`, respectively.

B.1.1 Special Relations: `isA`, `hasA`, `specializesTo`, and `partOf`

Some relations correspond to verbs with high frequencies in natural language: 'to be' and 'to have'. These are the relations ISA and HASA.

For instance: `isA(dog,animal)` expresses that a dog is a special kind of animal, and `hasA(dog,tail)` states that a dog is the possessor of a tail.

The relation `isA` occurs in two flavors. We can say `isA(dog,mammal)`, where both `dog` and `mammal` refer to collections of animals. Compare this to `isA(Pluto,dog)`, where `dog` is a collection with multiple animals, but `Pluto` refers to a single dog. The latter case is sometimes calls INSTANTIATION: we say that `Pluto` is a (concrete) instantiation of the class `dog`. The word 'concrete' here suggests that in `isA(Pluto,dog)`, `Pluto` is concrete whereas `dog` is abstract. We come back to concrete and abstract in **??**.

Both `isA` and `hasA` have opposites. These are SPECIALIZESTO and PARTOF. The same examples: `specializesTo(animal,dog)` and `partOf(tail,dog)`.

B.2 Constructing Concepts from Properties

Let us look at constructing concepts from properties. We use an example: a conceptual model of a four color ballpoint First we define a type, `fourCPen` as follows:

$$\texttt{fourCPen} = \texttt{[red:pen, green:pen, blue:pen, black:pen, use:}\{\texttt{TRUE, FALSE}\}\texttt{]}. \quad (B.1)$$

Notice that we have not yet defined `pen`. It must be some type since it occurs after ':'. For the property `use` (indicating if we can use a four color ballpoint to write - that is, if it contains ink and if a pen is out) we do know the type: it is the set {TRUE, FALSE}, also known as `Boolean`. So even if the type `pen` would be a singleton, the entire form between square brackets cannot be a singleton because both

$$\texttt{[red:pen, green:pen, blue:pen, black:pen, use:TRUE]:fourCPen} \quad (B.2)$$

and

$$\texttt{[red:pen, green:pen, blue:pen, black:pen, use:FALSE]:fourCPen}, \quad (B.3)$$

so `fourCPen` has at least two instances.

Next, notice the '=' instead of the ':', immediately after `fourCPen` in **??**. This indicates that `fourCPen` *equals* the expression in square brackets. With ':' instead, it would mean that `fourCPen` is a property with type `[red:pen, green:pen, blue:pen, black:pen, use:`{`TRUE, FALSE`}`]`.

Despite that the concept pen is not defined yet, we can form a new concept `myFourCPen:fourCPen`. Using the dot-notation explained in depth in Section **??**, we can refer to the pen called `red` in `myFourCPen` as `myRedPen=myFourCPen.red`.

Now we define the concept `pen`. It shall include the ink level, `level` and whether the pen is in or out, `iO`. This suggests

$$\texttt{pen=[level:}\{\texttt{0...100}\}\texttt{\%, iO:}\{\texttt{in,out}\}\texttt{].} \tag{B.4}$$

So, pen is a set, containing elements such as

$$\texttt{halfFullPen=[level:}\{\texttt{50}\}\texttt{\%, iO:}\{\texttt{in}\}\texttt{],} \tag{B.5}$$

abbreviated as

$$\texttt{[level:50\%, iO:in],} \tag{B.6}$$

(a set with a single element, $\{x\}$ will be abbreviated to x if no risk for confusion exists), and

$$\texttt{almostEmptyPen=[level:2\%, iO:out].} \tag{B.7}$$

These latter two concepts represent, respectively, a pen that is withdrawn ('in'), half full, and a nearly empty pen that is exposed ('out'). Both are singletons.

B.3 Extending Concepts: Inheritance

Concepts can be used to construct concepts with more detailed information. For instance

$$\texttt{myHalfFullPen:[level:50\%, iO:in]} \tag{B.8}$$

where

$$\begin{array}{r}\texttt{myHalfFullPen=}\\ \texttt{[level:50\%, iO:in, oxidation:}\{\texttt{rusty, clean, corroded}\}\texttt{].}\end{array} \tag{B.9}$$

The concept `myHalfFullPen` is a pen that is withdrawn, half full, which may or may not be corroded. Although `[level:50%, iO:in]` is a singleton, the concept `myHalfFullPen` is a set with three elements, for instance reflecting three stages in the history of my pen when it gradually got from a clean, via a corroded one, to a rusty pen.

Forming a concept from an existing concept by adding one or more properties is called INHERITANCE.

Inheritance can occur by adding one or more properties to a concept, as in the `myHalfFullPen` example. A second way is, by constraining the range of values for one or more properties. For instance, the concept `car` may have a property `color` with type `allColors`, where the concept `whiteCar` is a car that is white:

`isA(whiteCar,car)` (B.10)

where

`whiteCar.color=white`, (B.11)

and `white` is a subset of `allColors`.

Inheritance is discussed further in Section **??**.

Constraining the sets of values of properties of an inheriting concept may cause them all to become singletons. Then the resulting concept is also a singleton. Similar to `halfFullPen:pen`, `halfFullPen` being a singleton, inheriting from `pen`.

Inheritance leading to a singleton is called INSTANTIATION. Remember, though, that concepts are always sets - either with multiple elements or with a single element (singleton).

B.3.1 Aggregation: Arrays

Concepts are bundles of properties. Properties are distinguished by their names. Some names are as trivial as 0,1,2,... In that case, we may write `[0:concept0, 1:concept1, 2:concept2, ...]` or simply `[concept0, concept1, concept2]`. This is called an ARRAY.

An array behaves the same as any other concept, that is: the value (sets) of all of the properties can be arbitrary concepts. The concepts aggregated in an array are commonly called ' ELEMENTS' rather than 'properties'.

So there are two ways of bundling or AGGREGATION. To bundle the concepts `Tom`, `Dick` and `Harry`, we can form a concept with named properties as

`brotherHood=[brother1:Tom, brother2:Dick, brother3:Harry]`, (B.12)

or an array with numbered elements as

`otherHood=[Tom, Dick, Harry]`. (B.13)

To distinguish concepts such as `Tom` and his brothers from mere strings, like 'Tom', 'Dick', or 'Harry', we write

`brother=[name:string, age:integer]`, (B.14)

and

`Tom=[name:'Tom', age:74]` (B.15)

where

`Tom:brother`. (B.16)

So, literal strings are put in quotes; concepts are without quotes.

B.3.2 Denoting Properties of Concepts

To address a property of a concept, several notations are in use. Consider a concept `C`=[`P:vP, Q:vQ, R:vR`], a bundle of properties `P`, `Q` and `R` with respective values `vP`, `vQ`, `vR`. To address the value of property `P`, we can use one of the following notations:

dot notation `vP=C.P` . This notation is common in most programming languages. The dot can be interpreted as the relation `partOf`: `C.P` is `P`, seen as part of `C`. We have seen the dot notation before: we wrote `myFourCPen.red` to refer to the pen called `red` of `myFourCPen`.

index notation `vP=C[P]`. This notation follows if we compare a concept to an array. For example, the Dutch monarchy has had a number of Williams as kings. These were William I, William II, and William III. To bundle them, we define an array `William`, the elements are `William[I]`, `William[II]`, and `William[III]`. The values `I`, `II`, `III` are called INDICES in the array called `William`. Similarly, the players in a football team can be named as `PSV[1]`, `PSV[2]`, etc., being the first, second, ... player of PSV For concepts seen thus far, the properties have names rather than indices, but the way to address a property is identical.

function notation `vP=@(C,P)` allows compact expression in some cases. As follows: suppose that we can form a new concept by taking two concepts together - say, the addition of two vectors. Vectors have properties called `x`, `y` and `z`: let `v1=[x:1, y:2, z:3]` and `v2=[x:10, y:20, z:30]`, then `v1+v2=[x:11, y:22, z:33]`. If we are interested in the z-property of the sum of two vectors, the dot notation and the index notation don't work: where should we put the dot or the index? But writing `@(v1+v2,z)` is a compact alternative for either `v1.z+v2.z` or `v1[z]+v2[z]`.

subscript notation $\mathtt{vP}=\mathtt{P}_C$. This notation is very common in, for instance, physics. It stems from the pre-computer era: it uses as few as possible symbols, which is convenient to quickly write on a blackboard. It is not very useful, though, in contexts where computers are used: first, because a plain text editor has insufficient capability to render subscript symbols; further, the subscript notation is not standardized. Some people would write $\mathtt{P}^C$ or $\mathtt{C}_P$ instead of $\mathtt{P}_C$. Subscript notation is also a bit clumsy if the properties are indicated by numbers, such as indices in an array: $\mathtt{C}_{i+3}$ is more difficult to read than `C[i+3]` or `@(C,i+3)`. Further, if the value of property `P` is a concept in its own right, say with property `S`, the dot notation, index notation and function notation give a unique recipe (respectively `C.P.S`, `C[P][S]` and `@(@(C,P),S)`). The subscript notation would be something like $\mathtt{S}_{P_C}$, which soon becomes illegible.

The same example in all four notations:

$$\mathtt{myFourCPen.blue} = \mathtt{[level:60\%,iO:in]} \tag{B.17}$$
$$\mathtt{myFourCPen['blue']} = \mathtt{[level:60\%,iO:in]} \tag{B.18}$$
$$@(\mathtt{myFourCPen,blue}) = \mathtt{[level:60\%,iO:in]} \tag{B.19}$$
$$\mathtt{blue}_{\mathtt{myFourCPen}} = \mathtt{[level:60\%,iO:in]} \tag{B.20}$$

Notice that a property, when used as an index, is written in quotes.

B.4 `isA`, Abstractness and Concreteness

In connection to the `isA` (or `specializesAs`) relations, we may talk about ABSTRACT and CONCRETE. These terms are best understood in comparative sense. For two different concepts, `A` and `B`, if `isA(A,B)` then `B` is more abstract than `A` and `A` is more concrete than `B`. Indeed: `dog` is more concrete than `animal`, and `animal` is more abstract than `dog`. The term 'abstract' used without comparison (say, '`animal` is an abstract concept'), means that there is some other concept that is less abstract than `animal`. Similar, The term 'concrete' used without comparison (say, '`dog` is a concrete concept'), means that there is something else that is less concrete than `dog`.

With the notion of *property* we can give an alternative definition for the `isA` relation, as well as for the notions *abstract* and *concrete*. As follows:

If `isA(A,B)`, the following holds:

- every property of `B` also occurs as a property of `A`;
- for every `P`, property of `B`, the set of values of `A.P` is contained in the set of values of `B.P`.

For example: if there is a concept `dog` with property `weight` with a range between 0.0 and 100.0 kg, and the concept `Toby` such that `isA(Toby,dog)`, the concept `Toby` has also a property `weight` with values between 0.0 and 100.0.

We can now reformulate the definition of ' ABSTRACT' and ' CONCRETE':

`A` is more abstract than `B` if `B` has all properties in `A`, and for every property `A.P`, the values `B.P` are contained in the range of values for `A.P`.

`A` is more concrete than `B` if `A` has all properties in `B`, and for every property `B.P`, the values for `A.P` are contained in the range of values for `B.P`.

Furthermore, a concept is called concrete (not in comparative sense) if it has a uniquely determined value for each of its properties. 'Concrete' in this respect is always a provisional label. Indeed: any concrete concept, can be made into an abstract concept by adding one or more properties where these properties have more than one value. This happened when the concept `myHalfFullPen` was derived from `halfFullPen` in Section **??**.

Any concept that is not concrete, is abstract. Indeed, it can be made more concrete - either by narrowing the range of values for one its properties, or by adding a property with more than one value.

Remember that above we only talk about concepts - not about entities in the real world

For entities in the world, the assessment of 'abstract' or 'concrete' is more subtle.

B.4.1 Intension and Extension

We have now two sets of definitions of 'abstract' and 'concrete': one based on the `isA` relation, and one based on properties. They give rise to two distinct ways for defining concepts.

The first is called INTENSIONAL. This defines a concept by listing properties and their sets of values. It acknowledges that a concept is a bundle of properties. For instance: the concept `lantern` in a conceptual model could be defined as

$$\texttt{lantern=[height:\{0.5 ...12\}m,power:\{100,2000\}Watt].} \tag{B.21}$$

From an intensionally defined concept we can derive other intensionally derived concepts, using inheritance. For instance, a `streetLantern` is a `lantern` with an additional property `distanceToNextLantern`, with values between 5 and 50 (in meters). Or: the concept `lightSource` is an intensionally defined concept with a single property `power`, and `isA(lantern,lightSource)`.

The second way of defining concepts is called EXTENSIONAL. It works by explicit enumeration of all concepts that have a `partOf` relation to the defined concept. To define the concept `Beatles`, we write

```
beatles=[b1:PaulM, b2:JohnL, b3:GeorgeH, b4:RingoS].
```
(B.22)

So we construct a concept as a bundle of four properties with names `b1`, `b2`, `b3` and `b4`; the values of these four properties are the concepts `PaulM`, `JohnL`, `GeorgeH`, and `RingoS`, respectively.

To express that the Beatles can only be booked as a group and not as individual performers, the concept `beatles` should contain a property `groupAgenda:agenda`; the four individual properties `b1` ... `b4` no longer have a private property `myAgenda:agenda`.

Extensionally defined concepts can also be extended by inheritance. For instance, the concept `beatles` inherits from the concept `bandsWithPaul`= [`b1:PaulM`], but not from the concept `groupsWithPaul`= [`g1:PaulM`], since property `g1` does not occur in `beatles`. The concept `beatlesPlusGeorgeMartin`

An intensional definition can be seen as a constraint on the concepts that are implied. The extension, implied by an intensional definition, consists of all concepts that possess the desired properties, and that comply with these constraints. Extending an intensional definition, that is: adding a property or narrowing down the set of admitted values, we can never enlarge the extension. Perhaps we reduce the extension: concepts lacking the new property, or having values for some property beyond the tightened range no longer fall under its extension.

From the perspective of extension, the opposite holds: adding concepts to the extension, we may have to drop properties from the intensional definition; we may also have to relax restrictions on the admitted ranges of values for the properties.

So: increasing the intension will not increase the extension; increasing the extension will not increase the intension.

The smallest possible intension is the concept with no properties; the corresponding extension is the collection of all possible concepts. Indeed: a property corresponds to a constraint; without constraints for property `P`, *all* concepts having `P` are implied. Without properties, *all* concepts are implied. Conversely, the smallest possible extension is empty (the set containing no concepts at all), and all conceivable constraints hold for this set. (Example: all Spanish men, taller than 12 meter, have green hair and three arms. Each of them also has five arms: properties of an empty set may even contradict each other).

Balancing Intensions and Extensions

As an example we give an intensional definition of birds as

```
bird=[animal:TRUE, coveredWithFeathers:TRUE].
```
(B.23)

This implies a large extension with all living birds, but according to this definition, a plucked bird ceases to be a bird. Now suppose we add a property `canFly` with as value `TRUE`. As we know, kiwi, ostrich and penguin don't fly. If we find that these animals are nevertheless part of the extension, we must drop the constraint on the values for canFly. If, on the other hand, we adhere to the full intension, we must drop kiwi, ostrich and penguin from the extension: then these animals are no longer birds.

If we *do* think that flying is a relevant property for our modeling purpose, and we want to account for non-flying birds, we must do some additional structuring. We build two concepts, both inheriting from bird, namely: flyBird:bird and noFlyBird:bird, as follows:

flyBird=[animal:TRUE,coveredWithFeaters:TRUE,canFly:TRUE] (B.24)

and

noFlyBird=[animal:TRUE,coveredWithFeaters:TRUE,canFly:FALSE]. (B.25)

Inheritance helps structuring concepts. In Section **??** we study structuring of concepts more in depth.

Structuring Concepts to achieve Compactness

Building a conceptual model includes balancing extensions and intensions (extension='what do we want to include in our scope', and intension= 'what do we want to say about these concepts'). We strive for structure on our conceptual model. IMPOSE STRUCTURE means: introducing concepts that group or *aggregate* related concepts to obtain a simpler conceptual model.

Models that center around such structures often have a purpose to compress. In many cases, an intensional representation of a set is much more compact than an extensional one. For instance: 'even numbers between 0 and 1000', which is an intensional definition containing the properties evenOrOdd with value even, and bounds with value {0...1000}, is a more compact representation of [0,2,4,6,8,10,...,996,998,1000].

B.4.2 Ontologies; Taxonomies

A conceptual model, formulated in terms of properties and values for some domain is sometimes called an ONTOLOGY or TAXONOMY. Both terms refer roughly to the same thing: a structured collection of concepts with distinguishing properties and their values.

Properties in a taxonomy may apply to *all* concepts in that taxonomy. Then we call it an ORTHOGONAL taxonomy. Orthogonal taxonomies can be depicted as a table: rows are concepts, and every column corresponds to a property, and the values or value sets for the properties are in the cells of the table

In taxonomies that are not orthogonal, properties in one part of the taxonomy may differ from those in another part. Non-orthogonal taxonomies are typically HIERARCHICAL . This means that the entity-relation graph is a TREE. The example in Section **??** was a simple example, bird being the root of the tree and flyBird and noFlyBird its leaves.

Perhaps the oldest well-known example is Linnaeus' taxonomy of living beings. A living being is either a plant or an animal. This introduces three concepts: livingBeing, plant, animal. These are connected by the isA relation: both isA(plant,livingBeing) and isA(animal,livingBeing). From the concept animal, there are 6 specializations, that is: 6

inheriting concepts called `phylum`. One of them being `vertebrate` (having a segmented, bony spine). This specializes further into the 4 concepts (called `class` in biology) `mammal`, `amphibian`, `reptile`, `bird`, etc.. A property such as `clovenHoofed` (Dutch: 'evenhoevig'), which is `TRUE` for `deer` and `cow`, but `FALSE` for `horse`, does not apply to, say, `fish`, `insect` or `worm`

We give a simple example of a hierarchy where we list both the concepts, the properties and the values these take. We start with the set of concepts {`motorCycle, bus, bicycle, kanoo, cruiseShip, parachute, plane`}. The properties we use are `motorized`, abbreviated by `m` with values {`true,false`}; `environment`, abbreviated `e` with values {`air, water, land`}, and `passengers`, abbreviated `p` with values {`one, multiple`}. First, this gives

```
motorCycle = [m:true, e:land, p:one]
       bus = [m:true, e:land, p:multiple]
   bicycle = [m:false, e:land, p:one]
     kanoo = [m:false, e:water, p:one]
cruiseShip = [m:true, e:water, p:multiple]
 parachute = [m:false, e:air, p:one]
     plane = [m:true, e:air, p:multiple].
```

Next we introduce the abstract concept `motorizedLandVehicle` with `isA(motorCycle, motorizedLandVehicle)` and `isA(bus, motorizedLandVehicle)`. So

```
motorizedLandVehicle = [m:true, e:land, p:{one, multiple}].
```

Next we introduce the abstract concept `landVehicle` with `isA(bicycle, landVehicle)` and `isA(motorizedLandVehicle, landVehicle)`. So

```
landVehicle = [m:{true, false}, e:land, p:{one, multiple}].
```

Next we introduce the abstract concept `airVehicle` with `isA(parachute, airVehicle)` and `isA(plane, airVehicle)`. So

```
airVehicle = [m:{true, false}, e:air, p:{one, multiple}].
```

Next we introduce the abstract concept `waterVehicle` with `isA(kanoo, waterVehicle)` and `isA(cruiseShip, waterVehicle)`. So

```
waterVehicle = [m:{true, false}, e:water, p:{one, multiple}].
```

Finally, we introduce the abstract concept `vehicle` with `isA(landVehicle, vehicle)` and `isA(airVehicle, vehicle)`, and `isA(waterVehicle,vehicle)`. So

```
vehicle = [m:{true, false}, e:{land, water, air}, p:{one, multiple}].
```

This completes the hierarchy. It can be represented by a tree, such as in figure **??**.

The root node is `vehicle`.

The child nodes of `vehicle` are `landVehicle`, `airVehicle` and `waterVehicle`; these are distinguished by the values {`land`, `air`, `water`} of the property `e`.

The child nodes of `waterVehicle` are `kanoo` and `cruiseShip` that are *both* distinguished by the values {`one`, `multiple`} of the property `p`, *and* by the values {`true`, `false`} for the property `m`.

The child nodes of `airVehicle` are `parachute` and `plane` that are *both* distinguished by the values {`one`, `multiple`} of the property `p`, *and* by the values {`true`, `false`} for the property `m`.

Notice that both abstract concepts, `waterVehicle` and `airVehicle`, have the same distinguishing properties, namely `m` and `p`.

The child nodes of `landVehicle` are `bicycle` and `motorizedLandVehicle` that are distinguished by the values {`true`, `false`} of the property `m`. Notice that `p` is not a distinguishing property of `landVehicle`, since in both of its child nodes, `bicycle` and `motorizedLandVehicle`, the value `one` for property `p` occurs (namely: in `bicycle` and in `motorCycle`).

The child nodes of `motorizedLandVehicle` are `motorCycle` and `bus` that are distinguished only by the values {`one`, `multiple`} of the property `p`.

So apart from the leaf nodes (=nodes without children), `motorCycle, bus, bicycle, kanoo, cruiseShip, parachute, plane`, we had to introduce the non-leaf nodes `airVehicle, waterVehicle, landVehicle, motorizedLandVehicle` and the root node `vehicle`.

B.5 What is the Added Value of Formalizing Concepts?

It seems that working with relations such as `isA`, and terms like abstract, concrete, extension, intension, and inheritance is a complicated way to argue about trivial things. Why not simply resort to natural language?

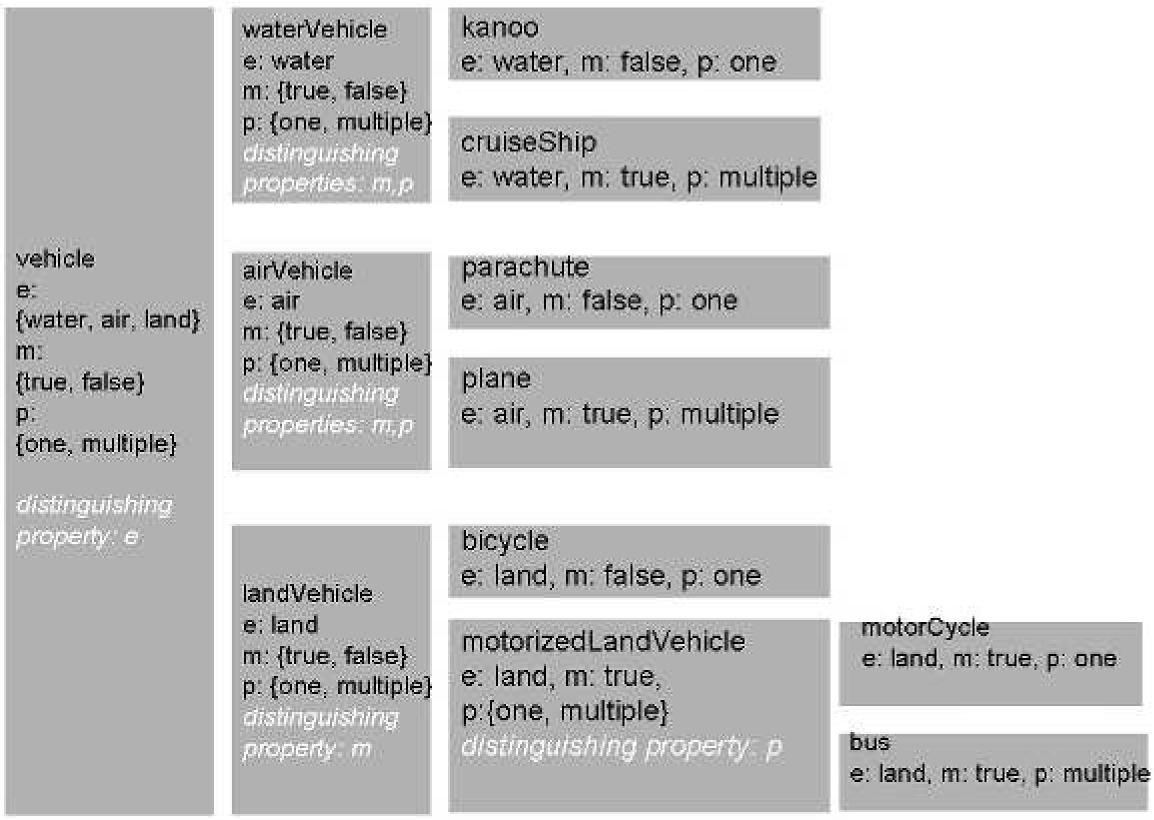
There are three reasons why conceptual modeling deserves a somewhat formal treatment.

- For non-trivial situations natural language soon becomes much too imprecise. Think of, for instance, solving problems with major industrial installations with many millions of components, interacting in billions of ways. A conceptual model, and hence a language to denote such models is a necessity to harness complexity.

- As of the 1990-ies, computers begin to be capable to support modeling - not only doing the calculations, but also verifying the consistency. An important step is the semantic web, also known as Web 2.0: a set of standards to write down documentation that is both understandable by humans and by computers. The notions we introduced here are all taken from the domain of semantic web. For this reason, the ideas from conceptual modeling as presented here are taken from computer science. Readers familiar with OBJECT ORIENTATION will recognize much of the terminology.

- Natural (that is, non-formal) language is not very well suited to describe the *organization* of information. With notions such as `isA`, `partOf` and inheritance we have simple yet powerful means to write down alternative versions of conceptual models so that, even before doing actual calculations, we may verify if the conceptual model indeed captures the essence of what we try to express.

  In particular, there are often several choices to which concept a property should belong. In the street lamp problem, for instance: the property `distanceToNextLantern` could be a property of a lantern, as in the conceptual model we constructed, but in hindsight it may be more convenient to make it a property of the concept `road`. Indeed, then the same concept `lantern` can figure in two instances of `road`, where in one they are closer together than in the other. If `distanceToNextLantern` is a property of `lantern`, we need to construct multiple `lantern`-concepts via inheritance, each with a different value for `distanceToNextLantern`.



Example of a hierarchy.

Figure B.1: Every node that is not a leaf node has two or more child nodes. These are distinguished by different values of one or more properties of the parent node. The latter properties are printed in white; these are called the distinguishing properties. (Illustration source: Kees van Overveld)

Notes, Index, and Glossary

Notes

[1]◁In a good approximation, planet's orbits are ellipses - at least, when viewed from a position that is fixed with respect to the sun. An ellips can be approximated by a circle, where the midpoint runs over another circle. In this respect, the Ptolemaic view was not unreasonable. When seen from the earth, however, planet's orbits are more complicated: they may contain loops and intersections. Ptolmy catered for these anomalies by having more and more advanced systems of circles running over circles, so called epicycloids

[2]◁Formulated as: 'A line joining a planet and the Sun sweeps out equal areas during equal intervals of time', This is the second of Keppler's three laws; the first stating the orbits are ellipses with the sun in a focal points; the third stating that 'The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit'.

[3]◁A consequence of the missing closed-form solution, however, is that it is not possible to prove that for instance the Solar System as we know it, is a stable configuration. So their is no mathematically rigorous proof that, say next year, the earth will still be in its familiar annual orbit. It is possible to run *numerical simulations*, though - and this of course has been done extensively. These show that there is no serious need for worry the first 100 million years or so - but the deeper question is: what is the amount of certainty that can be obtained from numerical simulations?

[4]◁It is interesting to think of what the criteria are for an answer to be acceptable in some given community.

[5]◁The distinction between a 'model' and a 'theory' is subtle. Most authors would call Newton's treatment of gravity a theory. It is usually understood that a theory has a wider scope of application than a model. So we speak of 'the theory of gravitation', from which 'a model for the Solar System' can be derived.

[6]◁Obviously, there is no end to possible 'why'-questions. One could (and physicist do) ask 'why does mass produce gravity?'. Interestingly, the communities who are satisfied by the answers to the increasingly fundamental questions rapidly get smaller.

[7]◁Communication is a process where a MESSAGE is conveyed from a sender to a receiver. When we view a user requirement document as a message, the sender is the prospect user, and the receiver is the designer.

[8]◁Exploration models may comprise of tables or catalogues. For instance, to aid the choice 'what sort of material should we choose for X', a catalogue of materials with their properties, offered by some vendor, or even the Periodic Table of Elements. Regarding this last example: it seems that the set of all chemical elements forms a closed set: we can enumerate all chemical elements. This may be true now, but it was certainly not the case when Mendelejev, the inventor of the Table, began his work. In fact, his exploration model helped to identify numerous as yet *undiscovered* chemical elements. So a model that is intended to aid exploration can serve to do predictions.

[9]◁This set of options can be COMPLETE or incomplete. For instance: if the decision is 'this rod should have some length X', or 'how long should this rod be', the set of possible outcomes is fully known: it will be some positive number of millimeters. In this case we are deciding with a complete set of options. An incomplete set could be associated with the question 'from what material do we make this rod': perhaps glass, carbon fibre or ceramic do not occur on our list simply because we have forgotten them, perhaps because we skipped the stage of doing a systematic exploration.

[10]◁Formal models contrast with INFORMAL MODELS. Until the 17th Century, mathematical notation was not standardized and little developed. Mathematical reasoning and non-mathematical reasoning, such as philosophical debates, roughly used the same type of language. Recent times see an increasing degree of rigor in mathematical vernacular, to the extent where computers verify mathematical argumentation. But even today there is room for THOUGHT EXPERIMENTS, most often formulated in natural language. Thought experiments, not expressed in any formal language such as mathematics, classify as informal models. In these lecture notes, our main focus will be on formal models.

[11]◁We don't take relativity theory into account here.

[12]◁We give an example where both continuous and discrete modeling takes place in one application. Consider a game of billiard. We want to develop a model to help a billiard player choosing the right cue position and angle to launch a bal such that the other two balls will be hit. We want to approach this problem by means of a dynamic SIMULATION - that is, we want to model the motions of the balls and the collisions as these take place in time. The set of collisions in one billiard shot form a discrete set of events. Each of them is determined by the initial state of the two colliding balls (that is, their position, velocity and rotational velocity). All these quantities are continuous, and the state after the collision follows from evaluating a mathematical function taking the initial states as input (see Appendix **??**). This yields a new set of values, and the balls will continue to roll on the billiard table after the collision. One problem, however, is to find out *where* a next collision will take place. There are two possible approaches here.

1. We ignore friction and rolling, and assume uniform motion:

   $$s(t) = s_0 + v \times t. \tag{B.26}$$

   Here, s and v are, respectively, a location and a vector in 2D space. Using formulas of this form for all three balls, we can compute if two balls, i and j will collide. This gives rise to the following condition:

   $$||s_{i,0} + v_i \times t - s_{j,0} + v_j \times t||^2 = (2\rho)^2, \tag{B.27}$$

   where ρ is the radius of a ball. The above formula simply asserts that the squared distance between the centers of two balls equals the squared sum of the radii - in other words: two balls touch. If this equation has a solution for some t, this is the time point of collision of the two balls i and j. Following this approach we work with continuous quantities only, and we don't need to sample anything.

2. The second approach is that we sample time t. That is, we replace the continuous quantity by a series of discrete values $t_1,\ t_2,\ t_3, \cdots$ where $t_{i+1} = t_i + \Delta t$ for some small Δt. In every new time step we recompute the location of all the three balls, and we check if there is a state where two balls are close enough to have touched. It turns out that this approach is in some respect simpler, in particular if we want to take friction and rolling into account. Sampling introduces a problem, however: since t only takes the discrete values, it is only possible to detect that two balls are close enough to collide if this occurs at one of the t_i. Suppose Δt was chosen too large: we then run the risk that we miss the event of a collision. On the other hand, choosing Δt very small is a sure waste of effort: we know beforehand that most of the tiny simulated displacements of the balls will show no collision.

The above dilemma is characteristic in virtual all modeling situations where sampling is used. When the sampling distance, sometimes called STEP SIZE, is too small we waste computational effort. If the step size is too large, however, it may introduce artifacts (such as missing crucial events like the collision in the above example).

For this reason, two modern trends in modeling are:

1. Use ADAPTIVE step sizes. Step sizes only need to be small in areas where the calculated quantity varies rapidly. In 'boring' regions stepsize can be big.

2. Using hybrid approaches, that is: mixing continuum calculations and sampling in one model.

[13]◁Wolfram's Mathematica, see http://www.wolfram.com, is a famous exception.

[14]◁The reader may wish to consider this table (again) after having read Section **??**

[15]◁Intuitively, two items can be close together, or far apart. This can be expressed in a number: 0 means that two items coincide, and the larger the number, the further they are apart. If items have coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), an expression such as Pythagoras' theorem, $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ follows this intuition. We could have chosen other expressions as well - each with their own properties. The current choice, for instance, has the property of being the same after arbitrary rotation of the configuration of the points in space.

[16]◁Numbers, operations, and sets are examples of mathematical objects. Mathematical objects are formal CONSTRUCTS. That is: they are ideas, invented by man, to help working with intuitive notions. Indeed, the mathematical objects we introduced here are all closely related to our intuitive feelings about space; they help to talk in a more precise way of what we experience when we perceive the space around us.

[17]◁Notice that that a deterministic model not necessarily gives correct answers: if we ignore the copper coins when assessing the current contents of our wallet, we know that our finding is an underestimate - but we know this with absolute certainty. Further, inaccuracies may result from non-perfect measurements of quantities that occur in the modeled system.

[18]◁Increasing an ensemble size will not always reduce the magnitude of fluctuations. No matter how often we throw a fair die, the fluctuations will always stay between 1 and 6, with equal chance for each outcome. But repeating a measurement of each time the same quantity (say, repeatedly weighing a given object) will make the average deviation decrease proportional to $1/\sqrt{n}$ with n measurements. Fluctuations cannot get arbitrary small, though: for fundamental physical reasons (quantum mechanics) there are lower limits to the achievable accuracy in any measurement.

[19]◁The amount of molecules in one mole is roughly 6×10^{23}, Avogadro's number, and at 273K, 1 mole of gas under 1 atmosphere occupies about 22 liter.

[20]◁Emergent behavior not always results from lumping: sometimes the items in the system that together form the emergent behavior are not as similar as molecules in a gas. For instance: the fact that a riding bicycle, despite its two wheels, does not topple over is an *emerging* result of the interplay of many forces, torques, and momenta. But since the various forces and torques all apply to different kinds of parts of the bicycle, we would not call the bicycle's emergent stability a bulk-property of the bicycle.

[21]◁It turns out that they do

[22]◁The existence of a causal mechanism seems to be related to the amount of compressibility of data. Data that cannot be compressed appears random, and there can be no mechanism inside something that is random. On the other hand, data that compresses extremely well - such as a long series of all equal numbers - also seem to contain at best a trivial mechanism. Apparently, interesting mechanisms prevail in these regime where there is some, but not too much compressibility in data. The existence of a non-trivial mechanism is sometimes seen as an indication for the presence of *meaning*.

[23]◁This is a slightly extended version from the modeling process as described in Edwards and Hamson.

[24]◁It is tempting to believe that such data exists outside the context of any model. This is generally not the case, however. Raw data is always the result of a measurement procedure. To complicate things, this is even true for

sensorial data. What we think to perceive with 'unarmed eye' is a construct, produced by millions of visual receptors in our retina and several hundreds of millions neurons performing aggregations and other processing thereafter. The only thing that we know for certain is that all this processing leaves its traces in the produced measurement result; we can merely wonder if there is something there that could be called 'the original' signal that triggers the entire cascade of transformations.

A measurement can be a physical process such as a thermometer measuring the temperature of the liquid in a bottle; it can also be a social process such as a questionnaire asking interviewees their appreciation of a new brand X of lemonade. Such measurements, however, only can be said to produce, respectively, 'the temperature of the liquid' or 'the average preference for lemonade X' if we believe the model that underlies this procedure. Let us focus on measuring temperature first. What we call '39.5°C' is the level of Mercury (Hg) half way between two marks on a glass tube, one mark at 39 and the next mark at 40. For this Hg level to signify something else than *just* a level of Hg in a glass tube, we need an *explanatory model* that links the construct 'temperature' (=that which we want to know) to the motion of molecules, and next the motion of molecules to the volume of an amount of Hg. This model involves a large number of assumptions, many of which are known to be only rough approximations. Rather we should say that the number '39.5' is generated by a procedure (involving, in this case, a physical instrument), and that perhaps, by reasoning 'backwards' through a model for this procedure, we may find a construct 'temperature' that has some meaning. For the questionnaire this is even clearer: every social scientist knows that the answers given in a street interview are determined, among other things, by the weather, the looks and attitude of the interviewer, the phrasing of the questions and hundreds of other factors. So: it requires significant modeling - and hence: assuming and believing a lot of implausible things - to explain that there exists this construct of 'preference for lemonade X', *independent* from the measurement that is used to assess it. 'Obtaining data' therefore is not merely 'obtaining data': incorporating a procedure of measurement forms a significant part of the modeling process.

[25]◁This is similar to the impossibility to assess if the lock on your door is sufficient. You can only empirically conclude the opposite: if somebody breaks in it was obviously *not* good enough. This observation generalizes to the attempts to experimentally assess the truth of a hypothesis: this is logically impossible. The best thing to do is to seriously attempt to invalidate the hypothesis. The longer it withstands serious attempts of invalidation, the more confidence it gains in communities of practitioners.

[26]◁To adjust the view direction, the mouse cursor should be dragged (=moved, holding the mouse button down) in the image. When doing so, a circle appears. If the cursor movements stay outside the circle, rotation takes place around the axis, perpendicular onto the image. It is then as if the image plane is grabbed by the mouse, and re-oriented. If dragging takes place inside the circle, however, the view is re-oriented as if one is re-orienting a ball by dragging a point on the surface of the ball.

[27]◁This demo uses the following method to adjust the view: the three sliders labeled 'yaw', 'roll' and 'pitch' can be used to rotate the view in 3D.

[28]◁According to some authors, the Indians did know that horses were animals. Most likely, *some* Indians had this knowledge, whereas others were ignorant of the fact.

[29]◁For an entity that is represented in a conceptual model we have a corresponding concept. Often, we refer to the entity by using the name of the concept. An entity that has *no* representation in any conceptual model has no distinct name. As soon as we talk about some entity, we give it a name, and we implicitly build a conceptual model for that entity: namely, a language fragment featuring that entity. So we can talk about, say, a table, but it is questionable if we really refer to a thing that is beyond any conceptual model. Some philosophers believe that anything that we can talk about is a concept in our own, temporary, conceptual model of the world, made out of language constructs. Even merely thinking about a table is likely to make words appear in your mind. And for pictorial thinking, one might say that a mental picture also is a concept in some conceptual model, and not the real table. Therefore things that are not concepts are, at best, extremely elusive.

[30]◁According to Genesis, the first thing Adam was allowed to do after he settled in the freshly installed Paradise was to give *names* to all the animals and plants he saw. In the process, he gave names to *groups* or *classes* of animals on the basis of their intrinsic appearance. He did not, for instance, called animals walking to the left the

leftWalkers to distinguish them from rightWalkers. Indeed, such distinction would be meaningless, as individual animals would constantly shift from being one concept to another. Also, he did not give names to the individual lions (Simba, Clarence, ...), but rather named them as a species: lion. For animals that became domesticated, however, he may have preferred individual names: the cow named Clara III used to produce more milk, but she was more recalcitrant when being milked, than Clara IV, and for the milkmaid it was good to know this difference. Names - and words in general - seem to be related to *purpose*.

[31]◁For this reason, the act of naming, in Christian tradition, is considered to be an essential, or symbolic act. Hence the protocol of baptizing, where the name-giving is lifted to the spiritual level.

[32]◁To distinguish concepts from entities, we will use a different font. Entities are written in the standard font, whereas concept, as well as other formal items, are written in this font.

[33]◁Formally, P(C) and C['P'] mean the same thing, provided that C is an aggregation possessing a property P. In computer implementation, however, the first version will only work if there is a function defined with the name P, returning C['P'] when called with argument C. With this function, we can retrieve the P-value for any concept that possesses a property P

[34]◁There is some loss of information, though. By omitting the particles 'a' and 'the', we can no longer distinguish 'a dog is near a lantern' from 'the dog is near the lantern' from 'the dog is near a lantern' from 'a dog is near the lantern'. Notice how the meaning of these four statements shifts.

[35]◁It is a good habit to work with concepts that are singular (lantern in stead of lanterns). That there may be a multitude of lanterns in the conceptual model will come later.

[36]◁The set of real numbers in mathematics is denoted by the symbol $\mathbb{R}$. Computers prefer standard letters, hence real. The issue of to what extent a computer can actually represent a real number is subtle, and is not discussed here.

[37]◁In some cases, we are more detailed, and establish not only relations between concepts, but also between *properties of* concepts.

[38]◁In practice, it may suffice just to report relations following from our immediate understanding of the modeled situation. On the other hand, there also may be relations that involve 3 or more things.

[39]◁This, of course, is only true if there is no first and no last lantern

[40]◁Cf. Appendix **??**, this is indeed a form of abstraction. In this view, 'quantity' is an abstract class, and properties inherit from this class.

[41]◁Here, wood etc. are concepts. For many purposes, however, we don't need to know the properties of these concepts. This is similar to the use of the word 'wood' in natural language without explicitly mentioning the sort of wood, the color, the density, etc.

[42]◁Provided that the square of the perimeter is at least 8 times the area. Indeed, not every pair of values for area and perimeter define a valid rectangle.

[43]◁http:en.wikipedia.orgwikiMohs_scale_of_mineral_hardness.

[44]◁Scales themselves are also ordered: operations allowed to a nominal scale are allowed to a partially ordered

scale; operations allowed to a partially ordered scale are allowed to a totally ordered scale; operations allowed to a totally ordered scale are allowed to an interval scale, and operations allowed to an interval scale are allowed to a ratio scale.

45◁Compare this to computing the average of two telephone numbers to find out the telephone number of somebody living in between.

46◁This table is a simplified version of the table in http://www.graphpad.com/faq/viewfaq.cfm?faq=1089.

47◁'Constant' here means: not depending on what is measured.

48◁Equivalence is a relation that is *reflexive*, *symmetric* and *transitive*. We encountered transitivity before when we discussed ordering in Section 2.6.2. The contstant-ratio-relation for units is transitive: if units u_1 and u_2 have a constant ratio, and so do u_2 and u_3, then u_1 and u_3 also have a constant ratio. 'REFLEXIVE', means that the relation applies between an item and itself. An example is `hasSameFatherAs`. The constant-ratio-relation for units is indeed reflexive: the ratio between a unit and itself is 1, which is constant. A relation is SYMMETRIC if the relation between `A` and `B` also applies between `B` and `A`. An example of a symmetric relation is `marriedTo`. The constant-ratio-relation for units is symmetric: if the ratio between u_1 and u_2 is constant (i.e., does not depend on what is being measured), then this holds as well for the reciprocal.

Concepts that are equivalent can be seen to belong to an EQUIVALENCE CLASS. Indeed, 'having a constant ratio to' is an equivalence relation. So there is an equivalence class consisting of all units such a m (meter), cm (centimeter), inch, km (kilometer), light year, etc. This equivalence class is called 'length'.

Similarly, units such as s (second), h (hour), d (day), μs (micro second), y (year), etc. are equivalent. They also form an equivalence class; this equivalence class is called 'time'.

49◁The fact that we write $\mathcal{L}$ and not $\mathcal{L}+\mathcal{L}+\mathcal{L}$ follows from reasoning by analogy with units. We write 3cm+5cm=8cm rather than 3cm+5cm=8(cm+cm) because this follows from applying the algebra both for the numbers and for the units, and the notation for dimensions follows from this rule.

50◁If two quantities are equal, their units need not to be equal: 3dm=30cm.

51◁Otherwise they would be the same dimension, by definition.

52◁Don't confuse the notation of a closed interval, $[a, b]$ with the aggregation of two non-named concepts or values, as in `[a,b]`. The first indicates a set of real numbers that typically has infinity many elements; it could also be denoted as $\{a...b\}$. The second is an ordered list with two elements, `a` and `b`.

53◁If c is an element of the domain of a single-variable function, it is a number. It may seem strange to call this 'a point'. Consider the following argument, however. Elements of the domain of a function of multiple variables correspond to multiple numbers, e.g. an element of the domain of the function $f(x, y) = x + y$ would be the pair $(3, 4)$. These form the coordinates of a two-dimensional point. Similar for functions of three variables (an element of the domain is a three-dimensional point), et cetera. Therefore it is common to call the elements of the domain of a function in general 'points' in stead of numbers - even if the function has only a single argument.

54◁There is a mathematical test that helps to determine if an extremum occurs at a critical point. This test uses second-order partial derivatives. The partial derivative of a function $f(x, y)$ again is a function of two variables. Therefore one can take partial derivatives with respect to x or y. Here we denote them by $f_{xx}(x, y)$, $f_{xy}(x, y)$, $f_{yx}(x, y)$ and $f_{yy}(x, y)$. If some conditions are satisfied one has $f_{xy}(x, y) = f_{yx}(x, y)$. Now we can formulate the *second derivative test*. Suppose that $f(x, y)$ has continuous second-order partial derivatives in some open disk containing the point (a, b) and that the first-order partial derivatives are zero in (a, b). Define the *discriminant* $D(a, b)$ for the point by $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$. The test gives: (i) If $D(a, b) > 0$ and

$f_{xx}(a,b) > 0$, then f has a local minimum at (a,b); (ii) If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then f has a local maximum at (a,b); (iii) If $D(a,b) < 0$, then f has a saddle point at (a,b); (iv) If $D(a,b) = 0$, then no conclusion can be drawn.

[55]◁To analyse the behavior of a function, we could of course just substitute different values for its arguments (=operate the sliders in ACCEL, specifying the values of all relevant quantities) and see what the calculated result is. A more efficient way is to go to the analysis tab in ACCEL and get a contour plot (click the checkbox 'graph'- this will force a contour plot instead of a graph). We want to visualize f (click 'f' in the right most table) in dependency of R1 and R2. Click the column 'H'(for horizontal) behind R1 and 'V'(for vertical) behind R2 in the leftmost table. The contourplot will be automatically drawn. All its properties, such as the range of the contour values, can be adjusted by filling in appropriate values in the text boxes. By moving the cursor in the figure, clicking, and dragging the cursor while holding the button down, the local values of R1, R2 and f are given.

[56]◁We clarify the notions of *closed* and *bounded*. A region $R \subset \mathbb{R}^2$ is BOUNDED if there is a disk that completely contains R. A region $R \subset \mathbb{R}^2$ is *closed* if it contains all its BOUNDARY POINTS. A point (a,b) is a boundary point of a region R if *every* open disk centered at (a,b) contains points in R and points outside R.

[57]◁More precisely stated: For an optimization problem, say to maximize $f(x,y)$, subject to inequality constraint $g(x,y) < 0$, a solution (x_S, y_S) is a point that satisfies $g(x_S, y_S) < 0$, whereas in other points (x_P, y_P) for which $g(x_P, y_P) < 0$, we have that $f(x_S, y_S) > f(x_P, y_P)$.

[58]◁The method of Lagrange multipliers can be used for optimization under constraints.

[59]◁This scheme is rather naive. It can be made more efficient in many ways. In particular, if the constrained inequalities are linear expressions in the unknowns, and if the function to be optimized is sufficiently well-behaved, there are powerful methods to solve the constrained optimization problem. These are the so-called SIMPLEX-METHODS .

[60]◁For further hints, consult Section **??**.

[61]◁Examples of state transitions, not relating to socks, are scoring a goal in soccer, having your birthday, or switching on the light.

[62]◁A PROCESS is a behavior, such as in physics, or a collection of behaviors, such as in computer science. There the word 'process' refers to 'a running program', which can show different behaviors, for instance in dependence of varying inputs

[63]◁There is a subtlety: in a collision of two billiard balls, energy, momentum and angular momentum are only conserved after the colliding balls both have assumed their new motion states. For this reason, in physical calculations, we sometimes act as if various properties change simultaneously.

[64]◁'Leaving out quantities' is not always a deliberate action on the side of the modeler. Often, quantities are left out where the modeler is ignorant that these quantities possibly could belong in the system. The modeled system behaves in a way that seems random: things happen without an observable cause. Consider the case of somebody getting sick - that is: suffering from certain symptoms. It is possible that this person contracted a virus some days earlier. Since the 'quantities' associated with the process of this virus multiplying are hidden, large parts of the process go unnoticed. Only by the time that 'exposed' quantities change their value (say, body temperature rising), the chain of causes and effects becomes manifest. Major parts of fundamental research in disciplines as varied as quantum physics and human biology are in constant search for the existence of HIDDEN QUANTITIES that may help explain the unexplained.

[65]◁We remember that *intervals* are not totally ordered. Therefore, if events would take a finite amount of time, we could not assume them to be totally ordered. Processes, taking a finite amount of time, for the same reason are partially ordered.

[66]◁The distinction between (external) events and (internal) transitions is not always clear-cut. To a large extent, it is determined by system borders, which may be arbitrary. For instance, consider somebody typing at a computer keyboard. If the typist and the keyboard are considered to be one system, all key strokes are internal transitions, caused by hidden properties, such as the brain states of the typing person. If the typing person is thought to be *outside* the system, the key strokes are external events. But assume that the keyboard is old and worn out, and at some point a key stroke causes it to break, producing some faulty signal to the computer. It requires quite a bit of detailed cause-and-effect reasoning to find out what, in this case, causes the fault: is it the typist's action (an external event), or is it the slowly degradation of the structural stability of the keyboard that reaches a state where it no longer can withstand the action of typing (an internal transition, taking place between values of one or more hidden quantities)? An interesting case study in this respect, with a complex interplay between external events and internal transitions is the analysis of the precise cause of the Harold of Free Enterprise disaster in 1987, see `http://rzv113.rz.tu-bs.de/Bieleschweig/pdfB2/deStefano_Bieleschweig.pdf`.

[67]◁The fact that transitions, and hence: states, are totally ordered is no sufficient condition, but it is at least a good indication. Indeed, if there would be transitions that *could* occur in arbitrary order (as with partially ordered time), these certainly could *not* be causally related, and then a functional dependency would not be possible.

[68]◁Instead of Equation 3.4 where i runs from 2 upwards, we could also write

$$Q_{i+1} = F_Q(Q_i, P_i, Q_{i-1}, P_{i-1}, \cdots), \tag{B.28}$$

with i starting at 1, or even

$$Q_{i+2} = F_Q(Q_{i+1}, P_{i+1}, Q_i, P_i, \cdots), \tag{B.29}$$

with i starting at 0. Indeed, i is merely a dummy quantity that can be offset by any convenient integer number. Expression B.28 and Expression B.29 have the same order if the difference between the index on the left hand side (in this case $i+1$ or $i+2$) and the lowest occurring value in the right hand side is the same.

[69]◁This resembles the way we calculate N factorial (written as $N!$), defined as

$$\begin{aligned} N! &= (N-1)! \times N \quad \text{if} \quad N > 0; \\ 0! &= 1. \end{aligned} \tag{B.30}$$

To find 5! we first need 4! and next multiply it with 5. To know 4! we must first know 3! etc., so we first calculate 0! which is 1; next 1!=1×0!, then 2!=2×1!, next 3!=3×2!, next 4!=4×3! and finally 5!=5×4!.

[70]◁ If we assume all Δ_i, s_i and g_i to be constant, say Δ, s and g, Expression 3.7 reduces to

$$A_{i+1} = A_i(1 - s\Delta) + g\Delta, \tag{B.31}$$
$$t_{i+1} = t_i + \Delta. \tag{B.32}$$

To calculate A for some t_t we need A_i for $i = t_t/\Delta$. Here we assume that t_t is an integer multiple of Δ. Next we use that Expression B.31 is an arithmetic - geometric series: for recursive relations

$$A_{i+1} = aA_i + g\Delta, \tag{B.33}$$

we have

$$A_i = a^i + \frac{a^{i-1} - 1}{a - 1} g\Delta, \tag{B.34}$$

which is easily verified by evaluating $aA_i + g\Delta$ and checking that this yields the expression for A_i with i replaced by $i+1$.

So, for $a = 1 - s\Delta$ we have

$$A_i = (1-s\Delta)^i + \frac{(1-s\Delta)^{i-1}-1}{-s\Delta} g\Delta. \tag{B.35}$$

We can use this result to calculate $A_{t_t/\Delta}$ for arbitrary t_t: we found a closed form solution for A_i for any t without unrolling the recursion. Closed form results not always exist, however. With little additional terms added to Expression B.31 we have to resort again to unrolling the recursion. For example: suppose that the bank gives an amount of interest ρ depending on the saved amount, $\rho_i = f_\rho(A_i)$. Then Expression B.31 changes into

$$A_{i+1} = A_i(1-s\Delta) + g\Delta + f_\rho(A_i), \tag{B.36}$$

and for most functions f_ρ there is no closed form solution for A_i, nor could we calculate the asymptotic value A_{infinity}. For *unrolling*, the addition of any extra terms is no complication: we just need to evaluate the recursive definition for the model quantities.

[71]◁This is the type of reasoning governmental agencies use when they decide on healthy national economies in the light of national debts etc.

[72]◁Such a step is sometimes called 'Ansatz', the German word for 'approach'. It refers to a heuristic procedure that often works, without guarantee for success.

[73]◁We have seen some cases where a recursive function gives an exact behavior of the motion of a point mass. There are many situations, however, where a recursive model does not hold exactly. Consider the case where two point masses are connected by a massless rod of length ρ. So the points stay at a fixed distance ρ from each other. They take the shape of a dumbbell. We write the recursive model for both:

$$\begin{aligned} r_{1;i+1} &= 2r_{1;i} - r_{1;i-1} + a_{1\to 2;i}\Delta^2; & \text{(B.37)}\\ r_{2;i+1} &= 2r_{2;i} - r_{2;i-1} + a_{2\to 1;i}\Delta^2. & \text{(B.38)} \end{aligned}$$

The accelerations $a_{1\to 2;i}$ and $a_{2\to 1;i}$ have a physical meaning. They result from the forces working between the two point masses. These are equal and opposite (action = - reaction), and they can vary in time (hence the index i). We don't know how big they are. We do know, however, that they are exactly big enough to keep the distance between the point masses constant and equal to ρ. Further, since they are equal and opposite, they must work along the line $r_{1;i} - r_{2;i}$. So we can replace the vectors $a_{1\to 2;i}$ and $a_{2\to 1;i}$ by a single unknown SCALAR quantity β:

$$\begin{aligned} r_{1;i+1} &= 2r_{1;i} - r_{1;i-1} + \beta(r_{1;i} - r_{2;i})\Delta^2; & \text{(B.39)}\\ r_{2;i+1} &= 2r_{2;i} - r_{2;i-1} + \beta(r_{2;i} - r_{1;i})\Delta^2. & \text{(B.40)} \end{aligned}$$

We are interested in the relative motion of the two point masses. Therefore we form the difference $R_i = r_{1;i} - r_{2;i}$. Subtracting Expression B.39 and Expression B.40 we get a recursive model for the relative movement:

$$\begin{aligned} R_{i+1} &= 2R_i - R_{i-1} + 2\beta R_i\Delta^2 & \text{(B.41)}\\ &= 2(1+\beta\Delta^2)R_i - R_{i-1} & \text{(B.42)} \end{aligned}$$

To find the value of β, we demand that $||R_{i+1}|| = ||R_{i-1}|| = ||R_i|| = \rho$, the distance between the point masses. To calculate $||x||$ we recall that $||x||^2 = (x,x)$, so:

$$\begin{aligned} \rho^2 &= ||R_{i+1}||^2\\ &= (R_{i+1}, R_{i+1})\\ &= (2(1+\beta\Delta^2)R_i - R_{i-1}, 2(1+\beta\Delta^2)R_i - R_{i-1})\\ &= 4(1+\beta\Delta^2)^2(R_i,R_i) + (R_{i-1},R_{i-1}) - 4(1+\beta\Delta^2)(R_i,R_{i-1})\\ &= 4(1+\beta\Delta^2)^2\rho^2 + \rho^2 - 4(1+\beta\Delta^2)(R_i,R_{i-1}). & \text{(B.43)} \end{aligned}$$

This reduces to

$$(1+\beta\Delta^2)^2\rho^2 = (1+\beta\Delta^2)(R_i, R_{i-1}). \tag{B.44}$$

Write $(R_i, R_{i-1}) = \rho^2\cos(\phi_\Delta)$, then ϕ_Δ is the rotation of the dumbbell over time lapse Δ.

Equation B.44 has two solutions. The trivial solution is

$$\beta = -\frac{1}{\Delta^2}. \tag{B.45}$$

If we substitute this back into Expression B.41, we get $R_{i+1} = -R_{i-1}$. This means that for any three subsequent states, $i-1$ and i and $i+1$, the orientation of R flips 180° from $i-1$ to $i+1$.

The other solution is $1+\beta\Delta^2 = \cos(\phi_\Delta)$. If Δ is small, which also means that ϕ_Δ is small, the cosine can be approximated by a Taylor series:

$$\cos(\phi_\Delta) = 1 - \phi_\Delta^2/2 + \phi_\Delta^4/4! - \cdots\,. \tag{B.46}$$

Since for small rotation ϕ_Δ is proportional to Δ, we write $\phi_\Delta = \omega\Delta$, where ω equals the current rotational velocity. Then, up to first order in Δ^2:

$$1+\beta\Delta^2 = 1 - \omega^2\Delta^2/2 + \omega^4\Delta^4/4! - \cdots\,, \tag{B.47}$$

or

$$\beta = -\frac{1}{2}\omega^2 + O(\Delta^2), \tag{B.48}$$

which is again a secondary school result: the centripetal force is proportional to the square of the rotational velocity.

This result is not exact. We make an error proportional to Δ^2. With sampling step size twice as small, the error gets 4 times as small.

We look again at the example of the rotating dumbbell. We started with a recursive model with time lapse Δ. If Δ is sufficiently small we see that the simulation for the relative motion is a uniform rotation. The rotation speed is closer to the secondary school result when Δ is smaller.

Moreover, we looked at the *relative* locations only. The absolute locations don't occur in Expression B.41. Any constant velocity can be added to $r_i - r_{i-1}$, for all i, and the simulation still holds. To check that the general solution for Expression B.37 consists of a rotation plus a uniform velocity we substitute

$$\begin{aligned} r_{1;i} &= r_0 + v_0\Delta i + \frac{1}{2}\rho\begin{pmatrix}\cos(\omega\Delta i)\\ \sin(\omega\Delta i)\end{pmatrix}, \\ r_{2;i} &= r_0 + v_0\Delta i - \frac{1}{2}\rho\begin{pmatrix}\cos(\omega\Delta i)\\ \sin(\omega\Delta i)\end{pmatrix} \end{aligned}$$

into Expression B.39 and Expression B.40; we easily verify that they hold for arbitrary v_0.

When sampling periodic phenomena, such as a rotating dumbbell, for given time lapse Δ, there is an upper limit to the frequency of the phenomenon that can be represented.

In this light we look again at Expression B.45. Suppose the dumbbell rotates increasingly faster, where the motion is sampled with constant Δ. When ω gets so large that, in between two subsequent transitions, the rotation is a full turn, this cannot be distinguished from a situation where it rotates two full turns, or three full turns, etc. This is also what happens e.g. in Western movies where rotating spoke wheels seem to rotate backwards. The sampling rate of 24 frames/second is not enough to capture the real rotation of the wheels.

In Expression B.48, β is only approximately proportional to ω^2 if $\omega^2\Delta^2$ is small compared to 1. This is a fundamental limitation to sampling. It is called ALIASING. Aliasing means that there is a phenomenon, periodic with frequency ν_p, that is sampled with a frequency ν_s. (Re)constructing the phenomenon from the samples works well if ν_s is sufficiently high compared to ν_p. If ν_s is too low, the reconstruction differs from the original phenomenon. It assumes an 'alternative identity', which is the literal meaning of the word 'alias'.

We study one more example of unrolling recursive definitions. We make a recursive model for a mass spring system. Again we start from the order-2 version. The force is proportional to the deviation from a rest position, r_0:

$$r_{i+1} = 2r_i - r_{i-1} - \omega^2\Delta^2(r_i - r_0), \tag{B.49}$$

where $\omega^2 = \frac{C}{m}$, C the spring constant, and m the mass.

We try a solution of the form

$$r_i = A\cos(\Delta\phi i) + r_0, \tag{B.50}$$

for unknown ϕ and A. So

$$\begin{aligned} r_{i+1} &= A(\cos(\Delta\phi i)\cos(\Delta\phi) - \sin(\Delta\phi i)\sin(\Delta\phi)) + r_0; & \text{(B.51)} \\ r_{i-1} &= A(\cos(\Delta\phi i)\cos(\Delta\phi) + \sin(\Delta\phi i)\sin(\Delta\phi)) + r_0. & \text{(B.52)} \end{aligned}$$

Form $r_{i+1} - 2r_i + r_{i-1}$ and equate this to $\omega^2\Delta^2(r_i - r_0)$. The result must hold for any i; this gives

$$2 - 2\cos\Delta\phi = \omega^2\Delta^2 \tag{B.53}$$

and arbitrary A. As with the rotating dumbbell, we find

$$\phi = \omega(1 + O(\Delta^2\phi^4\omega^{-2})). \tag{B.54}$$

We interpret this as follows. For Δ^2 small compared to $\omega^2\phi^{-4} = \frac{C}{m}^2\phi^{-4} \approx \frac{m}{C}^2$, that is, a spring with small spring constant C or large mass m, the solution oscillates with a frequency $\phi \approx \omega$. When Δ goes to 0, ϕ goes to the value ω from the high school physics result.

If, on the other hand, for a given Δ, the value $\frac{C}{m}$ is large, something else happens.

In the case of the dumbbell we had built in that the distance between the point masses stay constant. The simulation goes into a 'trivial' mode when Δ is too large. It then flips its orientation every two transitions.

In the mass-spring example there is no built-in mechanism that keeps the solution within bounds. The repeated evaluation of Expression 74, for $\omega\Delta$ too large, becomes UNSTABLE.

If a recursive model of the form $r_{i+1} = 2r_i - r_{i-1} + K\Delta^2$ gets unstable, the subsequent distances $||r_i - r_{i-1}||$ get increasingly larger. In other words, the kinetic energy of the moving point masses ($\frac{1}{2}mv^2$, or $\frac{1}{2}m||r_i - r_{i-1}||^2\Delta^{-2}$) gets increasingly larger. We can understand this as follows. For a system to be stable, its kinetic energy needs to be more or less constant. So any force should contribute, on average, no work to the system. The dot product $(K_i, r_i - r_{i-1})$, on average, should be zero, where K_i is the force in state i. For a spring, for instance, the point mass moves in the direction of the force for half of the time, and in the opposite direction for the other half. If Δ is too large, there will be evaluation errors in the recursive model. Then there is no reason that the calculated $(K_i, r_i - r_{i-1})$ also is zero on average. The simulated system acquires kinetic energy. Indeed: the kinetic energy can only increase, since it is quadratic in the velocity. So whether the error in the velocity is positive or negative, the error in the kinetic energy will always be positive. As a result, differences $r_i - r_{i-1}$ will be slightly too large, and kinetic energy increases even further. The errors reinforce each other, and soon the simulation goes 'out of hand'.

In physical reality, just the opposite occurs. If we have an oscillating mass spring system, after a while it slows down. This is because of friction or damping. There is always a force that works in *opposite direction* of the movement. In the absence of such force, we would have $||r_{i+1} - r_i|| = ||r_i - r_{i-1}||$ (according to Expression 3.13), and hence conservation of energy.

If, on the other hand, $||r_{i+1} - r_i|| = \sigma||r_i - r_{i-1}||$, with $0 < \sigma < 1$, velocities get increasingly smaller. A simple way to build in such DISSIPATION (=loss of energy in a dynamic system, typically due to friction or damping) is by introducing a force that always counteracts the movement. Since forces, and hence accelerations, can be added together, such a counteractive contribution can be simply added to other forces in the recursive model. So we write

$$r_{i+1} = 2r_i - r_i + \Delta^2K - \epsilon(r_i - r_{i-1}) \tag{B.55}$$

instead of

$$r_{i+1} = 2r_i - r_i + \Delta^2 K, \tag{B.56}$$

for whatever acceleration K we want to consider. Indeed, for $K = 0$ we get

$$r_{i+1} = 2r_i - r_i - \epsilon(r_i - r_{i-1}), \tag{B.57}$$

so

$$\begin{aligned} r_{i+1} - r_i &= r_i - r_{i-1} - \epsilon(r_i - r_{i-1}) \\ &= (1-\epsilon)(r_i - r_{i-1}). \end{aligned} \tag{B.58}$$

For small positive ϵ, in the absence of other forces, the kinetic energy decreases in time with a factor $(1-\epsilon)^2$ per time step. So adding a term $-\epsilon(r_i - r_{i-1})$ imitates the effect of dissipation in a physical system, and stabilizes our calculations. *Stabilizing* means: protecting against 'going out of hand'. The addition of a term $-\epsilon(r_i - r_{i-1})$ is trivial for the evaluation of the recursive functions in unrolling the simulation. Approaching this solution using differential equations (see Expression 3.3.4) is much more elaborate.

[74]◁To see the corresponding differential equation for a recurrent function of order 2, we realize that

$$r(t+\Delta) = r(t) + \Delta\frac{d}{dt}r(t) + \Delta^2\frac{d^2}{dt^2}r(t)/2! + \Delta^3\frac{d^3}{dt^3}r(t)/3! + O(\Delta^4) \tag{B.59}$$

$$r(t) = r(t) \tag{B.60}$$

$$r(t-\Delta) = r(t) - \Delta\frac{d}{dt}r(t) + \Delta^2\frac{d^2}{dt^2}r(t)/2! - \Delta^3\frac{d^3}{dt^3}r(t)/3! + O(\Delta^4). \tag{B.61}$$

By adding two times Expression B.60 and subtracting once Expression B.61 we get

$$\frac{d^2}{dt^2}r(t) = \frac{r(t+\Delta) - 2r(t) + r(t-\Delta)}{\Delta^2} + O(\Delta^4), \tag{B.62}$$

or

$$r(t+\Delta) = 2r(t) - r(t-\Delta) + \Delta^2\frac{d^2}{dt^2}r(t) + O(\Delta^4). \tag{B.63}$$

This helps to understand why we encountered the expression '$2r_i - r_{i-1}$' in Expression 3.22, Expression B.37 and Expression 74. In the form of differential equations, these models read, respectively:

$$\frac{d^2}{dt^2}r(t) = a \quad \text{(constant acceleration);} \tag{B.64}$$

$$\begin{aligned} \frac{d^2}{dt^2}r_1(t) &= a_{1\to 2}(t), \quad \text{(massless rod)} \\ \frac{d^2}{dt^2}r_2(t) &= a_{2\to 1}(t); \end{aligned} \tag{B.65}$$

and

$$\frac{d^2}{dt^2}r(t) = -\omega^2(r(t) - r_0) \quad \text{(mass spring system).} \tag{B.66}$$

As an illustration, we give the closed-form solution of the last one. We verify that $r(t) = r_0 + A\cos(\omega t)$ for arbitrary A solves Equation B.66. Similar as with recurrent functions, these differential equations have order two: named after the highest occurring derivative in the right hand side.

When we introduced dissipation in the recurrent functions, we added a term $-\epsilon(r_i - r_{i-1})$. In the case where $\Delta \to 0$, the difference $r_i - r_{i-1}$ plays the role of $\Delta \frac{d}{dt} r(t)$. This is inconvenient, since Δ goes to 0. But we remember that the interpretation of ϵ was a reduction of speed *per time step*. So we still can work with a dissipation term $-\epsilon \frac{d}{dt} r(t)$, but the dimension of ϵ has to be 1/time. Indeed, the dissipation term needs to be proportional to 1/time^2 in order to add it to other acceleration terms.

Including dissipation the mass spring system becomes

$$\frac{d^2}{dt^2} r(t) = -\omega^2 (r(t) - r_0) - \epsilon \frac{d}{dt} r(t) \quad \text{(dampened mass spring system)}. \tag{B.67}$$

This illustrates our remark about the advantages and disadvantages of using differential equations versus recurrent functions. Adding dissipation causes no additional complication for unrolling the recursion. The solution for the combined 1st- and 2nd order differential equation Expression B.67, however, is harder to guess than the solution for Expression B.66. A closed-form solution is possible; a full treatment of linear second order differential equations, however, falls outside the scope of these lecture notes.

[75]◁Some differential equations can be solved in closed form. That means that the solution has the form of a function rather than a list of numbers. We have seen an example of a closed-form solution: in the budget example, we got an expression for the asymptotic value for A in case of a constant income and a constant spending fraction (Expression 3.9). For modeling purposes such as *analysis*, a closed form is preferable.

Dynamical models in the form of differential equations may help us to learn something about the solution without having to unroll a simulation. Similar as with the budget example, however, a warning is in place. Adding a non-linear term or otherwise changing a solvable differential equation, perhaps only marginally, may make it unsolvable with analytic techniques. (For linear differential equations with a sufficiently small non-linear additional term, there is a technique called LINEARIZATION, which boils down to doing a first-order Taylor expansion to the non-linear contribution). This is in contrast with unrolling the recursive version. As long as the recursive function can be evaluated, it doesn't matter what form it has. The risk with numerical evaluation, on the other hand, is the threat of large unaccuracy or even instability if Δ is too large.

We look at the connection between recursive functions and differential equations. Let us first consider a recursive function, where r_i is the unknown quantity, and p_i are given quantities.

$$r_i = F_r(r_{i-1}, p_{i-1}). \tag{B.68}$$

With $t = \Delta i$, this is equivalent to $r(t+\Delta) = F_r(r(t), p(t))$. In the limit for $\Delta \to 0$, we can expand

$$r(t+\Delta) = r(t) + \Delta \frac{d}{dt} r(t) + O(\Delta^2) = F_r(r(t), p(t)). \tag{B.69}$$

This assumes that r, as a function of time t, behaves well. 'Well behaving' means: r first needs to be defined in any t, in a sufficiently narrow surrounding of t it should should not vary more than proportional to the size of the surrounding, and, most importantly, its derivative should exist. If this condition is met, we write

$$\frac{d}{dt} r(t) = \lim_{\Delta \to 0} \frac{F_r(r(t), p(t)) - r(t)}{\Delta}, \tag{B.70}$$

showing the correspondence between the derivative $\frac{d}{dt} r(t)$ and the recursive function F_r. The differential equations corresponding to Expression 3.5 and Expression 3.10 can be written as

$$\begin{aligned}
A_{i+1} &= A_i(1 - s_i \Delta_i) + g_i \Delta_i \\
&= A_i - s_i \Delta_i A_i + g_i \Delta_i \qquad \text{so} \\
A_{i+1} - A_i &= g_i \Delta_i - s_i \Delta_i A_i \qquad \text{or} \\
\frac{A_{i+1} - A_i}{\Delta_i} &= g_i - s_i A_i; \qquad \text{with } \Delta_i \to 0: \\
\frac{d}{dt} A(t) &= g(t) - s(t) A(t),
\end{aligned} \tag{B.71}$$

and $r_{i+1} = r_i + v\Delta$ or, equivalently $\frac{d}{dt} r(t) = v(t)$.

Index

hasA◁ (heeftEen)
relation that expresses that a concept owns a second concept;
Example: hasA(dog,tail) states that a dog is the possessor of a tail , 130

isA
]◁ (isEen)
relation that expresses that a concept is a specialization of a second concept;
Example: isA(dog,animal) expresses that a dog is a special kind of animal , 130

partOf◁ (deelVan)
relation that expresses that a concept is a sub-concept in a second concept;
Example: partOf(tail,dog) , 130

specializesTo◁ (specialiseertTot)
relation that expresses that a concept is a more general instance of a second concept;
Example: specializesTo(animal, dog) , 130

a priori◁ (a priori, van tevoren)
from the start, before doing anything;
Example: even before weighing a glass of water, you know that the found weight will be larger than the weight of the glass when empty , 24

abstract◁ (abstract, generiek)
-definition in terms of properties: of two concepts, A and B, A is more abstract than B if B has all properties in A, and for all properties in A, the values for B are within the range of those for A. 'Abstract' (not in comparative sense) means: being not concrete;
(no example) , 134

from two different concepts A and B, where isA(A,B), B is called more abstract than A;
Example: isA(dog,animal), and animal is more abstract than dog. 'A is abstract' (not in comparative sense) means that there is something else that is less abstract than A. So saying that A is abstract means that there is a B such that isA(B,A) , 134

abstraction◁ (abstractie)
- as a purpose of models: the process of leaving out details that are unnecessary for some given purpose;
Example: by leaving out the size and shape, both gas molecules and cars can be abstracted to moving points. This enables treating the flow of gas and the flow of traffic in the same way , 11

adaptive◁ (aanpassend, zelflerend)
the property of some quantity that it takes an adequate value, given the actual circumstance;
Example: an adaptive step size in sampling means, that it gets smaller if calculated or measured values in subsequent samples differ more; if these values differ little, the step size gets larger , 142

aggregate◁ (samenvoegen)
grouping together;
Example: the bundle of properties with their values in a concept are aggregated , 54

aggregation◁ (samenvatting)
obtaining data that are characteristic for an entire ensemble from data associated to the individual members of that ensemble;
Example: consider a population with

individuals that can be assumed similar, that each have a salary. Taking the average of all salaries produces the average salary, which is a property of the population. Averaging is an example of aggregating , 27

aggregation◁ (samenvatting)
bundling of concepts into one concept, either as named properties in a generic concept, or as numbered elements in an array;
Example: `Tom`, `Dick`, `Harry` can be aggregated into an array as `[Tom, Dick, Harry]`, or as `[brother1:Tom, brother2:Dick, brother3:Harry]` , 132

algorithm◁ (algoritme, rekenrecept)
programmed sequence of mathematical instructions, typically performed by a computer;
Example: long division (Dutch: 'staartdeling') is an algorithm to obtain the quotient of two numbers , 19

aliasing◁ (aliasing)
the aspect of sampling that, for a given sampling constant (e.g., a time lapse for sampling in time) there is an upper limit to the amount of detail in the sampled signal that can be reproduced;
Example: audio pitches of close to 44100 Hz can not be reproduced on an audio CD; rotating spoke wheels in movies may seem to turn backwards; near horizontal lines on a raster image show jaggies , 150

analysis◁ (analyse)
- as a purpose for models: attempting to find out about certain properties of a system, not by studying the system proper, but a model of the system instead;
Example: after a traffic accident, it is common to draw a schematic sketch of the situation. This is a model of the accident, used to analyse which party was responsible for the accident , 12

analytic◁ (analytisch)
- of a modeling strategy: making use of formal mathematical elaborations, operating on symbols rather than on numbers. Opposite to numerical;
Example: finding the maximum of a differentiable function by setting the first derivative equal to 0 , 19

AND◁ (en)
logical operator: P AND Q is true only if P and Q are true;
Example: a transaction is a purchase if both an item of value has been transferred from A to B and a sum of money has been transferred from B to A , 24

angle◁ (hoek, boog)
the distance between directions;
Example: a perpendicular angle is 90 degrees or pi/2 radians , 20

approximation order◁ (benaderingsorde)
a numerical approximation has order n if halving the sampling step size gives a reduction of the error of 1/2 to the power n;
Example: estimating the area underneath a function using the rectangle rule, approximating the function as piecewise constant, has an error that is proportional to the width of the piecewise constant segments to the 3rd power: it is order 3 , 110

arc◁ (boog, kant, zijde)
an element in a graph, next to node;
Example: in a diagram consisting of boxes and arrows, the arrows are the (directed) arcs; the boxes are the nodes. Indeed, arcs can be directed or undirected: a directed arc is an arrow , 26

argument◁ (argument)
-of a function: the value, taken from the domain of the function, that serves as input;
Example: for the function 'color', an argument could be anything colored , 56

arity◁ (ariteit)
- as property of relations: the way to express how many concepts are engaged

in the relation;
Example: a monogamous marriage is a 1-1 relation; a polygamous marriage is 1-n (1 male and multiple female partners); a polyandric marriage is n-1 (one female and multiple male partners); a hippy-commune is an n-m marriage (multiple male and female partners) , 129

array◁ (lijst, reeks, rij)
a concept where the property names are subsequent integers;
Example: `beatles=[John, Paul, George, Ringo]` is a way to denote the Beatles as a concept, where `John = beatles[0]`, etc. , 132

arrow◁ (pijl)
directed arc in a graph;
Example: if a relation, say `isA` is represented by an arrow, the opposite arrow represents `specializesTo` , 26

artifact◁ (kunstproduct, maaksel)
something made by man, as opposed to a natural object;
Example: an artefact needs not to be material: symphonies, laws and organizations are artifacts , 12

aspect ratio◁ (aspect(verhouding))
the ratio between height and width of a shape (e.g., a rectangle);
Example: a square has aspect ratio 1, same as a circle , 67

assumption◁ (veronderstelling, aanname)
a non-proven, perhaps even false proposition about the state of affairs that is taken to be true in order for further propositions to be deducible;
Example: in a system involving geometric optics, a mirror is assumed to be ideally planar. Material objects can never be mathematically planar, but in some contexts it can be reasonable to ignore deviations from planarity , 28

asymptotic behavior◁ (asymptotisch gedrag, langetermijngedrag)
behavior in the long run;
Example: for a pendulum with length l, the oscillation period is proportional to the square root of l, provided that the amplitude is small enough. But due to friction, any free swinging pendulum, in the long run, will reach a state where this condition is fulfilled. So the asymptotic value of the oscillation period of a free swinging pendulum is constant , 106

asynchronous◁ (ongerelateerd in tijd, uit de maat)
of an event with respect to some process P: the event can occur at any time during the sequence of actions in P. Opposite of synchronous;
Example: an incoming telephone call will typically be asynchronous with whatever we are doing , 96

ATBD◁ (-)
artifact to be designed;
Example: a novel type of mobile phone that should double as electric razor , 12

atomic◁ (atomair, ondeelbaar)
-of a value or a type: does not consist of a (bundle) of multiple properties;
Example: a number, a boolean, or a string are examples of values that need no further information to be fully known , 59

averaging-out◁ (uitmiddelen)
the effect that, in a sufficiently large ensemble of similar entities, individual variations can sometimes be ignored in comparison with average values;
Example: if we throw a fair die, the number of times we throw each of the possible outcomes 1,2,...,6 will all approach to 16.66...percent - even though they never will get exactly the same. Their variations get smaller with an increasing size of the ensemble (=the total number of repetitions of the experiment of throwing the dice) , 22

behavior◁ (gedraging)
a route through a state space;
Example: Hamlet's part in Shakespeare's play is the only known

route through this Danish prince's state space , 91

binding◁ (binding)
the relation between a value and a property, assuming that value;
Example: at the time of writing, the value '56' is bound to the property 'age' of the concept 'author'. In a few months, the bound value will be '57' , 89

binding◁ (binding)
the association of a value to a quantity (say, a property of a concept);
Example: the value '54', at the time of writing, is bound to the property age of the concept `authorOfThisText` , 27

black box◁ (zwarte doos)
a form of modeling where no claims about causal mechanisms are made; the model is obtained by compressing the observable information of some system;
Example: many models in biology, psychology, medicine and economics are black box models because the inner working of the modeled systems are too complex to represent , 25

Boolean◁ (Boolean)
the type with values `TRUE,FALSE`;
Example: the value of the expression 'it is currently raining' is `TRUE` or `FALSE`; therefore this expression (and every other proposition) has type Boolean , 62

boundary points◁ (randpunten)
a point (a, b) is a boundary point of a region R if *every* open disk centered at (a, b) contains points in R and points outside R;
(no example) , 147

bounded◁ (begrensd)
a region $R \subset \mathbb{R}^2$ is bounded if there is a disk that completely contains R;
(no example) , 147

bulk quantity◁ (bulk-eigenschappen)
a quantity of a system, consisting of many similar entities, where some form of aggregation applies;
Example: pressure and temperature of a given amount of gas , 22

calculate◁ (rekenen, berekenen)
obtain the resulting value from a formal expression by applying rules from arithmetic or calculus; opposed to reasoning;
Example: obtaining the volume of a rectangular box by multiplying its height, width and depth , 24

calculus◁ (calculus)
the part of mathematics involving functions, limits, differentials and integrals;
Example: a Taylor expansion of a sufficiently differentiable function is a device from calculus , 29

characteristic time◁ (karakteristieke tijd)
in a dynamical process, an amount of time needed to perform a typical (part) of the behavior;
Example: for a period dynamical process, the period is a characteristic time. For a behavior of exponential increase or decrease, the characteristic time is the amount of time needed for doubling or halving , 116

closed interval or disk◁ (gesloten interval of gebied)
an interval (or disk in two dimensions) is closed if it contains all its boundary points;
(no example) , 74

closed◁ (gesloten)
- of a set of values: it is possible to enumerate all values, either directly by listing them all, or indirectly by giving a finite recipe to generate them. Opposite to open;
Example: enumerating all values: 'the taste of this candy can be sweet, sour, salt or bitter'; generating the values: 'the shape of a cog wheel is a circle with two or more equal shaped indentations, placed at regular intervals on the perimeter' , 13

coefficient◁ (coëfficiënt)
a quantity, often occurring as a factor that is multiplied with a variable;
Example: in the function z = ax+by+c,

a is the coefficient for x, and b is the coefficient for y , 62
communication◁ (communicatie)
- as a purpose of models: a way to inform some intended audience about what is modeled;
Example: a list of numbers, representing the outcome of an experiment can be a means to communicate this outcome to an interested reader , 13
complete◁ (volledig)
- of a set of options: including all possible outcomes;
Example: `earth`, `water`, `fire`, `air` is the complete set of concepts that can be obtained by combining values `hot`, `cold` for property `temperature` and values `wet`, `dry` for property `humidity` , 142
compound◁ (samengesteld)
- of a value or a type: consisting of (a bundle of) multiple properties, each with their own value;
Example: a vector, consisting of 2 (in 2D) or 3 (in 3D) coordinates is a compound value. A compound value is a concept in its own right. , 59
concept graph◁ (concept(en) graaf)
entity relation graph;
(no example) , 61
concept◁ (idee, voorstelling)
defined as a bundle of properties;
(no example) , 54
concept◁ (idee, voorstelling)
a mentally conceived or imagined entity, used in a model and representing some entity in the modeled system;
(no example) , 52
conceptualization◁ (conceptualisatie)
stage in the modeling process, comprising of building the conceptual model and choosing quantities;
(no example) , 26
conclusion◁ (conclusie, afronding)
stage in the modeling process, comprising of presenting and interpreting the result;
(no example) , 29
concrete◁ (concreet, specifiek)
-definition in terms of properties: of two concepts, `A` and `B`, `A` is more concrete than `B` if `A` has all properties in `B`, and for all properties in `B`, the values for `A` are within the range of those for `B`. 'Concrete' (not in comparative sense) means: 'having only properties with a unique defined value';
(no example) , 134
from two different concepts `A` and `B`, where `isA(A,B)`, `A` is called more concrete than `B`;
Example: `isA(dog,animal)`, and `dog` is more concrete than `animal`. '`A` is concrete' (not in comparative sense) means that there is something else that is more abstract than `A`. So saying that `A` is concrete means that there is a `B` such that `isA(A,B)` , 134
congruent◁ (gelijkvormig)
of two geometric figures: have the same shape, that is: one geometric figure can be mapped onto the other one using just rotation, scaling and translation;
Example: all circles are congruent; all equilateral triangles are congruent , 63
consistent◁ (samenhangend, kloppend)
such that no contradiction results ;
Example: a collection of statements (propositions) is contradictive if it is possible to deduce both a statement and its negation. See contradiction , 63
constant◁ (constante)
a quantity with a value that does not change;
Example: many so-called physical constants (e.g., the speed of light, the mass of an electron) are assumed to have an invariant value , 42
constraint◁ (beperking)
a limitation that applies to values for properties of a concept. See contingent;
Example: for the area and perimeter of a rectangle the constraint holds that the square of the perimeter is at least 8 times the area. Constraints can be

(logically or mathematically) necessary, as in this example; they can also be contingent, for instance the constraint that something should not be heavier than X kg because otherwise it falls through the floor , 64

construct◁ (construct)
mental artifact, an abstract notion invented by man;
Example: mathematical objects, but also social notions such as 'marriage', 'possession', 'justice', ... , 143

context (of a problem)◁ (probleemcontext)
set of circumstances, events and conditions, not immediately part of a problem statement, that partially determine the success of the solution;
Example: for a model concerning the illumination of a motor way, the circumstance that the motor way is in an area with frequent fog is part of the problem context , 29

contingent◁ (contingent)
not logically or mathematically necessary. See constraint;
Example: as a mathematical necessity, for positive a, 3a is larger than 2a. In a shop, however, the fact that three products are more expensive than two is contingent: there may be a discount programme that sells 'three for the price of two' , 64

continuity◁ (continuïteit)
the property of a quantity to be able to assume all values between a minimum and maximum, without skipping even the tiniest hole;
Example: the speed of a material object can take a continuous set of values; the price of a good, to be payed in currency with a smallest coin can not take a continuous set of values , 17

continuous◁ (continu, aaneengesloten)
having the property of continuity;
(no example) , 17

continuous◁ (continu)
-of functions: $f(x)$ is continuous in $x = c$ if, for any $\epsilon > 0$, however small, we can find a $\delta > 0$ such that there is an x, $c - \delta < x < c + \delta$ with $f(c) - \epsilon < f(x) < f(c) + \epsilon$;
Example: $f(x) = 3x + 5$ is continuous in $x = 4$. Indeed, choose $\delta = \epsilon/3$, then $4 - \delta < x < 4 + \delta$ implies that $17 - \epsilon < 3x + 5 < 17 + \epsilon$, 74

contour plot◁ (contourenkaart)
visualisation of a function of two variables by means of one or more level curves;
Example: weather maps often show contour plots to indicate locations of equal temperature (isotherms) or locations of equal pressure (isobars) , 43

contour◁ (contour)
level curve;
(no example) , 43

contradiction◁ (tegenspraak)
see consistent;
Example: the statements `greaterThan (a,b)`, `greaterThan(b,c)`, and `greaterThan(c,a)` together are contradictive: the latter two can be combined to deduce `greaterThan(b,a)`, which conflicts the first statement , 63

coordinate◁ (coördinaat)
quantity, used to distinguish spatial locations;
Example: in a cartesian system, coordinates are length, width, and height; in a polar system, coordinates are radial distance and angle; in a spherical system, coordinates are radial distance, azimuth and elevation , 19

coplanar◁ (co-planair)
being confined to the same plane;
Example: the left and right rail in a railroad track in flat terrain are coplanar , 20

correlated◁ (gecorreleerd, verwant)
two sets of quantities are correlated if one can help predict the other;
Example: there is a correlation between the amount of alcohol a woman consumes during pregnancy and the birth weight of the baby , 22

correspondence◁ (correspondentie, overeenkomst)
-a relation between two things;
Example: an entity (something in the modeled system) corresponds to a concept (something in the model). This form of correspondence is also called representation , 52

counting◁ (tellen)
establishing a correspondence between distinct entities in a set and numbers 1,2,3 ... The number of entities is the largest number encountered;
Example: our number system is base ten because our ancestors used their fingers to make a correspondence between amounts of things and numbers. In French, 'quatre-vingt' for 80 reminds of a tradition where toes were used as well , 18

critical number, point◁ (kritieke waarde, punt)
an element of the domain of f is a critical number (one-dimensional domain) or a critical point (two or more-dimensional domain) if all derivatives are zero, or if some derivative is undefined;
Example: Both for the function $f(x, y) = x^2 + y^2$ and for the function $g(x, y) = 1/(x^2 + y^2)$, (0,0) is a critical point. For f, both partial derivatives in (0,0) are 0; for g, both partial derivatives in (0,0) are undefined , 74

data compression◁ (datacompressie)
- as a purpose of models: the representation of a body of information such that less space is needed;
Example: the information '0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30' can be compressed to 'positive even numbers less than 31' , 11

database◁ (databank, informatieopslag)
structure to hold information, typically in the form of tables of mathematical objects, suitable for representation in a computer;
Example: the list of ingredients in a meal, together with a list of prices for ingredients allows the calculation of the price for a meal , 24

deadlock◁ (impasse, patstelling)
in a dynamic process: a state that cannot lead to a following state;
Example: a computer system that is said to 'hang' often is in a state of deadlock , 95

decision◁ (beslissen)
- as a purpose of models: aid in taking a decision, either by optimization or by constraint satisfaction;
Example: the crucial steps in a design are the design decisions , 13

deduction◁ (afleiding)
logical inference in which the conclusion is of no greater generality than the premisse. See induction;
Example: if we observe 20 swans that are all white, we can deduce that it is possible that swans are white , 12

definition◁ (definitie)
- in modeling: first stage of a modeling process;
Example: the problem of 'how to illuminate a motor way' is defined more precisely by asking: 'is it possible to obtain sufficient illumination conditions with LEDs for this particular motor way' , 25

design◁ ((technologisch) ontwerp)
(as opposed to research) the process of systematically taking decisions with the primary purpose of creating value for stake holders;
Example: the decisions leading to the realization of a machine, a material, an organization etc. , 9

deterministic◁ (bepaald)
involving only known steps and dependencies;
Example: the outcome of a die throw is not deterministic, whereas the outcome of throwing a quarter in a functioning coffee machine is deterministic , 21

differential equation◁ (differentiaalvergelijking)
a mathematical equation where the un-

knowns are functions rather than quantities, and where derivatives of the unknown functions occur;
Example: Newton's motion equation, F=ma is a differential equation since acceleration a is the second derivative of the location with respect to time , 111

dimension◁ (dimensie)
-in organizing information: property, or aspect that can help distinguishing individual items;
Example: 'gender', 'age' and 'educational level' are three possible dimensions in demography to distinguish individuals in a population , 14
an equivalence class, belonging to the equivalence relation 'has a constant ratio with' between units;
Example: length, time, force, energy, etc. , 71

direction◁ (richting)
that which two different parallel lines have in common;
Example: North, South, East and West are distinct directions, defined everywhere on the globe except on the North- and Southpole , 20

discrete◁ (discreet, telbaar)
-of a quantity: distinct; there is no smooth route to go from one value to another. Values of discrete quantities result from counting;
Example: states of a game of chess form a discrete set , 18

disjoint◁ (losstaand)
separated from something else, standing alone;
Example: the legs of a table, although connected by the table top, are disjoint entities , 51

dissipation◁ ((wrijvings)verlies)
loss of energy in a dynamic system, typically as a result of friction or damping;
Example: a dashpot in a mass-spring system increases the rate of energy loss in the system, converting kinetic energy into heat , 151

distance◁ (afstand)
a measure for the proximity of two items;
Example: in spatial coordinates, it can e.g. be expressed as the square root of the sum of squares of the difference between the coordinates of the respective items. In a broader context, it can be applied to non-spatial quantities as well , 20

domain convention◁ (domein conventie)
for a given function $f(x, y)$ the largest set of pairs (x, y) for which this function $f(x, y)$ can be evaluated, unless the domain is explicitly given by a smaller set;
(no example) , 41

domain◁ (domein)
-of a function: the set of values the argument of a function can be taken from;
Example: for the function 'age', everything that was ever born or created , 56

dot-notation◁ (puntnotatie)
notation to address a property of a concept: for concept `C`, `C.P` is the value of property `P`;
(no example) , 133

dynamic◁ (dynamisch, tijdsafhankelijk)
involving time; opposite to static;
Example: the balance of forces that keep a dike from collapsing under wind and surf load , 17

edge◁ (boog, kant, zijde)
see arc;
(no example) , 26

element◁ (element)
-used, in stead of 'property', to refer to the concepts aggregated in an array;
Example: in the array `[red,grn,blu]`, `red`, `grn`, `blu` are its elements , 132

emergent◁ ((onverwacht) verschijnend)
-of a phenomenon: something that appears as the result of some internal process;
Example: the behavior of an individual ant is limited and well-understood. Yet a colony of ants is capable of building structures like anthills, which can not

readily be seen to result from combining individual ants' behaviors , 22
empirical◁ (empirisch)
based on observation - as opposed to 'found by reasoning' or 'resulting from a definition';
Example: results following from a laboratory experiment, a questionnaire, etc. , 10
ensemble◁ (ensemble, collectief)
a collection of many entities that each behave stochastically, but similarly, so that the law of large numbers helps obtaining meaningful expectation values;
Example: the molecules of a confined amount of gas in thermal equilibrium , 21
entity-relation graph◁ (entiteit-relatiegraaf)
a graph where nodes are concepts, referring to entities, and arcs are relations;
Example: a city map, an electronic circuit , 61
entity◁ (entiteit)
anything, represented by a concept, of which information is represented in the model;
Example: entities can be material or immaterial, real or virtual , 26
entity◁ (entiteit)
something in the modeled system (and not in the model) that can be referred to; something that can be distinguished from another entity;
Example: pointing to something can be a way to distinguish it from other things, even if these have no names: 'this' as opposed to 'that' , 52
equality◁ (gelijkheid)
relation between two quantities stating that they have equal values;
Example: x=y and x=5 implies that y=5 , 77
equidistant◁ (gelijk verdeeld)
-of a series of values: having the same distance between any two subsequent values;
Example: the pearls in a pearl necklace , 19
equilibrium◁ (evenwicht)
- state: the state of a model such that, when slightly perturbed, it will try to get back to the initial state;
Example: a spring, when gently pulled and released, after a short while will assume its initial length , 33
equivalence class◁ (equivalentieklasse)
collection of things that are pairwise connected by an equivalence relation;
Example: 'has the same color as' is an equivalence relation; an equivalence class with this relation is the class of red things; another equivalence class is the class of green things , 146
event◁ (gebeurtenis)
the external cause for a transition in a state chart;
Example: the telephone rings, somebody insert a coin in a coffee machine, or a billiard ball collides with another billiard ball , 98
execution◁ (uitvoering, executie)
stage in the modeling process, comprising of operating the model, obtaining a result plus an estimate of the accuracy of the result;
Example: after composing a set of equations to represent the behavior of a modeled system, the execution stage amounts to the solution of these equations by mathematical or numerical means , 28
exhaustive◁ (volledig, uitputtend)
complete, taking all possibilities or all options into account;
Example: simple board games like tic-tac-toe can be exhaustively analyzed: all possible states can easily be enumerated. This is not feasible for chess, checkers and most card games , 32
expert system◁ (expert systeem)
a system to represent expert knowledge in the form of a set of rules;
Example: a system to aid medical diagnosis could represent, for a number of pathological conditions, the observable symptoms. Since symptoms do

not match one-to-one to pathological conditions, an expert system needs the ability to reason with logical operations , 24

explanation◁ (uitleg)

- as a purpose of models: the association between two domains of knowledge, where this association to some communities may provide a sufficient answer to a 'why' question;
  Example: Q: 'why does the temperature of a cold object increase when it is brought into contact with a hot object?' A: 'because there is a substance, called phlogiston, flowing from warm to cold bodies'. One domain is the (by now abandoned) Medieval theory of phlogiston, the other domain is an experience or observation from daily practice , 10

exploration◁ (verkenning)

- as a purpose of models: imposing a structure on an open domain to facilitate producing the elements of the domain;
  Example: suppose the domain is 'planar shapes'. This domain is infinite (it contains circles, squares, star shapes, letters, ...), and there is no a priori manner to classify them. Exploring could be done by proposing properties on the domain (such as 'symmetry', 'size', 'curved or straight') that help the classification , 13

extensional◁ (extensioneel)

-of a concept definition: listing all concepts that engage in a `partOf` relation with the concept to be defined. Opposed to intensional;
  Example: an extensional definition of the concept `Beatles` would be: `partOf (PaulM, Beatles)`, `partOf (JohnL, Beatles)`, `partOf (GeorgeH, Beatles)`, `partOf (RingoS, Beatles)` , 135

extremal◁ (extreem)

-of an element of the domain of a function: the property of being an extremum;
  Example: For the function $f(x) = x^2$, $x = 0$ is extremal , 75

extremum◁ (extreem (punt))

an element in the domain of a function f where f assumes a local or global maximum or minimum;
  Example: The function $f(x) = x^2$ assumes an extremum in $x = 0$ - which is a global minimum , 75

factor◁ (factor)

a quantity, often occurring in a product expression. See also term;
  Example: 2, 3, and 5 are the prime factors of the number 60 because 2 x 2 x 3 x 5 = 60 , 62

feasible region◁ (geen vertaling)

part of the domain of a function, that is to be optimized, where constraints are fulfilled;
  Example: for the inequality constraints x¿0, y¿0 and x+y¡3, the feasible region is an isosceles triangle, aligned with the coordinate axes of the x-y plane , 79

fitting◁ (aanpassen)

obtaining the value of a quantity in a formula by demanding that this formula adequately compresses a set of data, for instance in black box modeling;
  Example: by fitting an exponential curve through a collection of radioactivity intensities, measured at regular time intervals, we can deduce the halftime of that radioactive material , 24

formal◁ (formeel)

expressed in terms of mathematical or logical formulas, or in terms of an algorithm;
  (no example) , 23

not relying on human interpretation;
  (no example) , 27

that which is defined within a logically consistent system, and does not require interpretation by human intelligence in order to be operated. Arithmetic is an example of a formal system;
  Example: 'when John was five years

younger than Suzy, Suzy was twice as old as John' can be formally expressed as y+5 = x and x = 2y; it follows that y = 5 and x = 10 which can be interpreted as the ages of John and Suzy at the time the riddle refers to , 27

formalization◁ (formalisering)
stage in the modeling process, comprising of obtaining values for quantities, and introducing mathematical relations between quantities;
Example: the translation from a wiring scheme (= an entity-relationship graph, hence a conceptual model) of two parallel resistors with values R1 and R2 to the formula
R = R1 R2 /(R1+R2) , 27

formula◁ (formule)
formal relation between quantities;
Example: F=ma for the relationship between a force F, and the acceleration a caused by that force on a point with mass m , 27

function notation◁ (functienotatie)
notation to address a property of a concept: for concept `C`, at-symbol`(C,P)` is the value of property `P`;
(no example) , 133

function of two variables◁ (functie van twee variabelen)
the function $f(x, y)$ is a rule that assigns a real number $f(x, y)$ to each ordered pair of real numbers (x, y) in the domain $D \subset \mathbb{R}^2$ of the function;
(no example) , 40

function, recursive◁ (recursieve functie)
a function where the calculation of the return value needs the evaluation of the same function on another argument value;
Example: a function that computes the sum of elements from a list, f(a1,a2,a3,...) may be defined as the first element of the list, a1, plus the same function applied to the remainder of the list; the sum over an empty list is zero. Recursive functions are convenient to represent quantities that are time dependent. For instance f(x,current)= x,previous+d express that the current value of quantity x depends on the previous value of the same quantity. If d is constant, this function describes a uniformly incrementing behavior , 101

function◁ (functie)
a prescription to produce a uniquely determined value, given a value;
Example: mathematical functions such as square, square root, etc., are familiar examples. But properties are also functions of the concept they belong to: the age of a person is a function of that person , 55

geometry◁ (meetkunde)
the part of mathematics studying the relations between mathematical objects, stemming from formalizing intuitions related to our perception of space;
Example: Pythagoras theorem is a result that can be proven by geometric means , 20

glass box◁ (glazen doos)
a form of modeling based on (assumed) causal mechanisms; the model is obtained by representing the causal relations by mathematical expressions;
Example: many models in chemistry and mechanical engineering are glass box models involving reaction mechanisms or laws of physics , 25

global maximum, absolute◁ (globaal maximum)
a value $f(c)$ (or $f(a, b)$ in two dimensions) is a global maximum on the region R if it is larger than (or equal to) all the other function values in the region R;
(no example) , 76

global minimum, absolute◁ (globaal minimum)
a value $f(c)$ (or $f(a, b)$ in two dimensions) is a global minimum on the region R if it is smaller than (or equal to) all the other function values in the region R;
(no example) , 76

graph (entity-relation)◁ (entiteiten-en-relaties graaf)

a diagram where nodes correspond to entities and (directed) arcs correspond to named relations;
Example: an electronic circuit, a structural formula in chemistry, or an annotated city map , 26

graph◁ (graaf, netwerk)
a drawing consisting of nodes and arcs;
Example: the London Underground map and the maps used by the Dutch Railways (NS) are graph representation of hundreds of kilometers of rail connections, stations and junctions , 26

grey box◁ (grijze doos)
- of a model: mix between a black box and a white (glass) box model;
Example: a model for propelling a ship using the theory of fluid flow (glass box), where coefficients for the friction between water and the ship hull are taken from measurements in a water tank (black box) , 25

ground truth◁ (vaststaand feit)
data that can be used to verify if the behavior of a model is consistent with our knowledge of the modeled system;
Example: a model for weather predictions can be run to predict yesterday's weather, using earlier weather data as input. The actual observations of yesterday's weather form ground truth, to compare the 'predictions' with , 32

hierarchy◁ (hiërarchie)
-example of;
Example: see figure **??** , 136

hierarchy◁ (hiërarchie)
a structure that can be depicted as a tree;
Example: the relation between a node and its parent is, for instance, `isA` or `descendsFrom` , 136

hypothesis◁ (hypothese, veronderstelling, vermoeden)
a postulated proposition or relation, that is assumed to be provisionally true, but that is subject to sceptic testing;
Example: a hypothesis in social science could be: 'frustration causes aggression'. In material science, a hypothesis could be that adding material X to a substance Y increases the melting temperature, etc. , 28

hypothetical◁ (hypothetisch, verondersteld)
imaginary;
Example: free electrons are hypothetical particles, transporting electricity through a conductor , 18

identity◁ (identiteit, eigenheid)
that which allows distinguishing one thing from other things;
Example: the identity of a Dutch citizen is reflected in his or her (unique) passport number; the chemical identity of an element from the periodic table is reflected in its atomic number , 52

IMPLIES◁ (impliceert, heeft als gevolg)
logical operator: P IMPLIES Q is true if Q is true or P is false;
Example: rain IMPLIES a wet ground; this means that either the ground is wet OR it is not raining , 24

independent◁ (onafhankelijk)
-of two quantities: one cannot be deduced from, or correlated with, the other;
Example: the yearly number of sunny days in Tokio and the yearly sales of bikinis in London are two independent functions of time , 63

index notation◁ (indexnotatie)
notation to address a property of a concept: for concept `C`, `C[P]` is the value of property `P`;
(no example) , 133

index◁ (index)
number to be written between [and], used to single out one element from an aggregation;
Example: For the aggregation p=[2,5,7], p[2] denotes 7, and the index is 2 , 55

index◁ (index)
quantity to select the desired element from an array;
Example: to obtain the third element

from an array x we write x[2] where '2' is an index , 133

individual◁ (individu, ondeelbare entiteit)
literally: that which cannot be divided;
Example: a human being in a population, or a molecule in a gas or liquid , 52

induction◁ (generalisatie)
a form of reasoning where an attempt is made to arrive at general conclusions from limited premisses. See deduction;
Example: if we observe 20 swans that are all white, we may be tempted to state that all swans are white - which is not necessarily true , 12

inequality◁ (ongelijkheid)
relation between two ordinal quantities stating that one is larger than the other;
Example: for two ranks in the army, p and q, p¿q and p=captain, q has a lower rank than captain. An inequality can be expressed with the relation ¡ or with the relation ¿: a¡b is equivalent to b¿a , 77

inference◁ (afleiding)
a formal operation with logic quantities and logic rules. See deduction;
Example: let P represent the proposition 'all balls are spherical', and Q the proposition 'a football is a ball', then the conclusion 'a football is spherical' can be drawn purely by using the logical structure of the propositions. If propositions are written down with sufficient precision, dedicated computer languages are capable to perform such deductive inference , 29

inheritance◁ (erfrelatie)
the construction of a new concept from an existing concept, either by adding one or more properties or by constraining the range of values for one or more properties;
Example: `policeCar` inherits from `car`, `smallDog` inherits from `dog` , 131

initial value◁ (beginwaarde)
a value to be provided for a quantity in a model for a dynamic system in order to calculate subsequent states;
Example: for a billiard shot, the velocity of the bal that was hit by the cue. For a system representing financial transactions: the initial amounts on the involved accounts , 103

instantiation◁ (instantiatie, instantie, voorbeeld)
application of the `isA` relation where the first argument is a singleton;
Example: `isA(Earth,planet)` , 130

instantiation◁ (instantie)
the construction of a concept, by constraining the sets of values of properties of a given concept, such that a singleton concept remains;
Example: in the relation `isA(myCar, car)`, the concept `myCar` is a singleton, assuming that I own only one car , 132

integer◁ (geheel (getal))
zero, or the successor of an integer;
Example: 3, 17, 888895, -4 , 18

intentional◁ (intensioneel)
-of a concept definition: listing a number of properties and their sets of values. Opposed to extensional;
Example: concept `lantern` is defined by the two properties `height` and `power`, both of type real, where height is between 0.5 meter and 12 meter, and power is between 100 and 5000 Watt , 134

interpolation◁ (interpolatie)
obtaining a value y, depending on a quantity t, from values y0 and y1, occurring for t-values t0 and t1, where t is between t0 and t1, as y = y0 + (y1-y0)(t-t0)/(t1-t0);
Example: John's length at age 12 was approximately the average of his length at ages 10 and 14, assuming growth rate to be constant in the period between 10 and 14 years , 18

interpretation◁ (duiding)
-as stage in the modeling process: the formulation of an answer to the initial problem in terms of the problem domain, rather than in terms of the model do-

main;
Example: imagine a model for the concentration of some medicine in the blood flow. The outcome of the model could be a table consisting of concentrations as a function of time. An interpretation could be: 'take two pills before breakfast and another pill just after lunch to have the fewest unwanted side effects' , 29

invariant◃ (invariant)
something that stays the same when circumstances, such as measurements, change;
Example: in measuring: any physical quantity stays the same if the laboratory moves with uniform velocity. Also: when measuring length l with a unit of length u, the value (l/u)u is invariant. Example not related to measuring: the ratio between perimeter and diameter of a circle stays the same if the circle is enlarged or reduced , 69

iso-(value) curve◃ (curve van gelijke waarden van een functie)
level curve;
(no example) , 43

iterate◃ (herhalen)
-of the modeling process: repeatedly go through the subsequent process stages, e.g. because of increasing understanding of the model's purpose;
Example: to obtain an increasingly better approximation of the square root of some S, one should start with an arbitrary positive number x and repeatedly replace x by (x+S/x)/2 , 31

iterative◃ (herhaaldelijk)
based on iteration;
Example: counting beans is an iterative process, where repeatedly we set apart one bean and increase a number N by one. When there are no more beans, N, started with zero, equals the number of beans , 19

knowledge base◃ (kennissysteem)
a database, especially designed to hold mathematical objects that represent propositions;
Example: a database with medical symptoms together with a database containing rules to link symptoms to diseases , 24

label◃ (label, naam, aanduiding)
- in a graph: a name or identity given to a node or an arc. In particular if arcs in a graph can have different meanings, these should be labeled;
Example: in a geographical map, the city names printed near dots representing cities are labels , 26

length◃ (lengte)
- of a curve: the number of sufficiently short line segments of unit length,
needed to approximate the shape of that curve;
Example: the length of a curve is found by having a hypothetical rope follow the curve; next stretching the rope and measuring the distance between its two end points , 20

level curve◃ (hoogtelijn)
a collection of points where a function of two variables takes the same values;
Example: if the function represents a temperature distribution over an area, the level curves are isotherms. Other examples are isobars (equal pressure) or iso-potential curves (equal electric potential) , 43

lifelock◃ (geen Nederlandse vertaling)
in a dynamic process, lifelock is a limited collection of states where no transitions exist that lead out of this collection;
Example: two polite people in front of a narrow passage, wanting to grant each other precedence ('after you-after you blocking') , 95

line◃ (lijn)
straight curve;
Example: the trajectory of a beam of light in a space with constant index of

refraction is a straight line , 20
linear approximation◁ (lineaire benadering)
approximation of a function, in the neighborhood of a given point of the domain, by a linear function;
Example: the linear approximation of the function $f(x, y)$ at the point (a, b) is defined as $L(x, y) = f_x(a, b)(x-a)+f_y(a, b)(y-b)+f(a, b)$, 45
linear◁ (lineair)
(1) to be described or approximated by an expression y = ax+b where x is an independent quantity and y is a dependent quantity; (2) the property, of a dependency y = f(x), that f(x1+x2) = f(x1)+f(x2) and f(sx) = sf(x), for all x, x1, x2 and real s;
Example: example of 1: the position of a point with constant velocity is a linear function of time; example of 2: the amount of heat, produced by burning gas, is linearly dependent of the amount of burned gas. See also superposition , 18
linearization◁ (linearisatie)
a technique to approximately solve non-linear differential equations where the nonlinear term is sufficiently small compared to linear terms;
(no example) , 153
local maximum, relative◁ (lokaal maximum)
a value $f(c)$ (or $f(a, b)$ in two dimensions) is a local maximum of f if it is larger than (or equal to) all the other function values in some open interval (or disk in two dimensions) containing c (or (a, b)) ;
(no example) , 74
local minimum, relative◁ (lokaal minimum)
a value $f(c)$ (or $f(a, b)$ in two dimensions) is a local minimum of f if it is smaller than (or equal to) all the other function values in some open interval (or disk in two dimensions) containing c (or (a, b)) ;
(no example) , 74
logistic growth◁ (logistische groei)
growth in the presence of limited resources;
Example: If bacteria would divide every minute, their amount would double every minute - growing beyond any bound. In practice, their growth rate decreases, among other things because of limited food, leading to a constant colony size , 117
lumping◁ (samenvoegen, klonteren)
the replacement of numerous quantities, each related to an individual item in a system by few quantities that apply to the system as a whole;
Example: molecules in a gas, citizen in a community, cars in a flow of traffic , 22

macro-irreversible◁ (macro onomkeerbaar)
the property of physical transformations that, when many degrees of freedom are involved, time order is not symmetric;
Example: friction always slows moving things down, turning motion energy into heat; a melting ice cube will turn in a little puddle of water, but a freezing, unconstrained puddle of water does not assume a cube shape , 87
mathematical object◁ (wiskundig object)
a concept used in mathematical reasoning;
Example: numbers, vectors, functions, matrices, geometric shapes , 20
median◁ (mediaan)
for an ordinal set: the largest value x such that the number of occurring values less than x is not larger than the number of occurring values larger than x;
Example: consider 5 sticks of lengths 1, 2, 5, 10, and 99 cm. Then 5 is the median value , 66
message◁ (boodschap)
finite amount of information, to be conveyed from a sender to a receiver;
Example: 'if you can read this you are driving too near by' , 141
micro-reversible◁ (micro omkeerbaar)
the property of physical interactions that, when few degrees of freedom are in-

volved, time order is symmetric;
Example: the collision between two ideal rigid spheres, the compression of an ideal spring , 87

model refinement◁ (model verfijning)
replacing (part of) a model by a more sophisticated version in order to better fulfill a purpose;
Example: in a geometric system, a spherical mirror can be first be assumed to reflect a parallel beam of light to one that converges through the focal point. This is only true for sufficiently narrow beams; model refinement may involve the more elaborate calculation of the actual shape of the reflected beam , 28

model, conceptual◁ (conceptueel model)
version of the model that comprises of concepts, their properties and relations between them, but not yet formal mathematical constructs. See model, formal;
Example: a block scheme for a chemical reactor is an example of an entity relationship model for that reactor , 26

model, formal◁ (formeel model)
version of the model that comprises of quantities and formal relations between them. See model, conceptual;
Example: the unknown currents flowing through a network of resistors, written as a set of linear equations , 27

model, formal◁ (formeel model)
model where mathematics, logic, or computer science plays a crucial role;
Example: next to performing experiments and the development of theories, doing simulations begins to be a third direction in various scientific disciplines. Simulations are an example of formal models , 16

model, immaterial◁ (model, immaterieel)
model, only involving information and information carriers (e.g., paper and ink, computer memory), and no other material objects;
Example: a mathematical model (e.g., functions and equations), a logical model (e.g., propositions and rules for deduction), or a software model (e.g., a simulation) , 16

model, informal◁ (model, informeel)
model where arguments and reasoning are mainly stated in natural language;
Example: Einstein's famous thought experiment that lead to the idea of special relativity started with the question 'what would happen if someone could travel on the front of a light ray'. Although some elementary mathematics is required to find the basic formula of special relativity, most part of the reasoning is informal , 142

model, material◁ (model, materieel)
model involving a material object;
Example: a scaled-down aircraft used in a wind tunnel to estimate aerodynamical properties of the original, full-scale version; guinea pig used in testing medicine to predict the reactions in humans to that medicine , 16

model, Monte Carlo◁ (Monte Carlo model)
kind of model where properties of a complex system are approximated by repeating stochastic calculations;
Example: the pressure of a gas, in dependence of temperature T, could be found from a simulation using sufficiently many elastically colliding point masses with well-defined initial kinetic energy, proportional to T. The name comes from the association between Monte Carlo and casino, hence stochastic processes , 22

model, purposes◁ (model)
-in relation to purpose. See purpose;
(no example) , 9

model◁ (model)
-definition: the (mental) construct resulting from going through the modeling process, to help fulfilling a purpose;
(no example) , 25

model◁ (model)
-steps common to all models;
Example: collecting data, performing

mathematical operation(s), interpreting the outcome of the mathematical operation(s) , 8
-various dimensions of models;
Example: static vs. dynamic, continuous vs. sampled vs. discrete, numerical vs. symbolic, geometric vs. non-geometric, deterministic vs. stochastic, calculating vs. reasoning, glass box vs. black box , 14
model◁ (model)
-as the result of a process;
Example: the stages in the modeling process are: definition, conceptualization, formalization, execution, and conclusion , 25
model◁ (model)
- as a means to achieve a purpose;
Example: doing a computer calculation on atmospheric data to predict the chance that tomorrow will bring rain , 7
modeled system◁ (gemodelleerd systeem)
the existing or hypothetical system to which a model relates;
Example: if the model is a geographic map, the modeled system is the depicted area on the surface of the Earth. If the model is a drawing of an electric circuit, consisting of rectangles and circles connected by lines, the modeled system is the physical assembly of resistors, capacitors and transistors to which this drawing relates. A modeled system is either a part of existing reality for models with a research purpose; it is part of not-yet existing, possible future reality for models with a design purpose , 25

node◁ (knoop, punt)
an element in a graph, next to arc;
Example: in a diagram consisting of boxes and arrows, the boxes are the nodes and the arrows are the arcs , 26
nominal◁ (nominaal)
of a set: a set that has no ordering;
Example: a set of countries, a set of plant species, a set of car brands, ... , 65
of quantities: the property that they cannot be ordered;
Example: materials, nationalities and tastes are nominal , 65
numerical◁ (numeriek)
operating on numbers instead of non-numerical symbols. Opposite to 'symbolic' or 'analytic';
Example: estimating the maximum of a function $y = f(x)$ on an x-interval by repeated evaluation in a series of closely spaced x-values, recording when the largest y value is obtained , 19

object orientation◁ (objectoriëntatie)
a flavor of conceptual modeling, based on the notion of interacting objects;
(no example) , 138
objective◁ (objectief)
does not depend on an individual observer, or on an individual opinion. Opposite of subjective;
Example: the viscosity of marshmallows is lower than the viscosity of caramel candy , 11
ontology◁ (ontologie, (model gebaseerd op) kennisleer)
-in conceptual modeling: a domain (=a collection of entities) described in terms of (shared) properties and their values;
Example: the ancient Greek identified four elements (the concepts `earth`, `air`, `water`, `fire`). An ontology on these four elements is often formed with properties `humidity`, with values `wet`, `dry`, `temperature`, with values `hot`, `cold`, as follows:
`fire=[temperature: hot, humidity: dry]`, etc.. Ontologies of this form can be depicted as tables , 136
ontology◁ (ontologie, (model gebaseerd op) kennisleer)
-as part of philosophy: the doctrine that

studies the things that are (Dutch: 'zijnsleer') - as opposed to epistemology: the study of what we can know (Dutch: 'kennisleer');
Example: an ontology of vehicles (train, bus, bicycle, ...) could use dimensions such as driving power (electricity, gasoline, biological effort, ...), purpose (transport goods, transport passengers), and operation (scheduled, individually operated, rented, ...) , 136

open disk◁ (open cirkelschijf)
the interior of a circle (i.e. all points inside in the circle but not on the circle;
(no example) , 74

open◁ (open)
- of a set of values: it is not possible to enumerate all values. Opposite to closed;
Example: 'the shape of this hole can be round or square or heart-shaped or something else ...' , 13

operation - mathematical◁ (bewerking, wiskundige)
processing of mathematical objects, such as numbers or functions, using mathematical or logical operators (such as add, subtract, differentiate, ...);
Example: $a(b+c) = ab+ac$ holds for arbitrary numbers a, b and c , 8

operation◁ (operatie, bewerking)
activity, such as evaluation, optimization, solving equations, numerical approximation, etc., to be performed with a model;
Example: a weather model is run with a set of empirical weather data as input to predict tomorrow's weather , 28

optimization◁ (optimalisatie)
- as purpose of a model: finding an answer to the question 'for which value(s) of quantity x is the resulting value of quantity y, depending on x, as good as possible', where 'good' needs to be specified ('large', 'small', ...);
Example: finding the shape of a tank such that the volume is as large as possible with a given wall surface area , 13

OR◁ (of)
logical operator: P OR Q is true if at least one of the two is true;
Example: precipitation is the weather condition where either rain OR snow falls from the sky , 24

ordening - partial◁ (partiële ordening)
a relation between elements from a set that is transitive, and that introduces no cycles;
Example: the relation `ancestorOf` for the collection of human beings , 65

ordening - total◁ (totale ordening, totale volgorde)
a relation between elements from a set that is transitive, anti-symmetric and total. 'Total' means that for any two different elements one exceeds the other;
Example: the relation `greaterThan()` for the collection of numbers , 65

order◁ (orde)
for the model of a dynamical system. This is the number of earlier states needed to evaluate the current state;
Example: for a system to represent financial transactions, the order is 1: to obtain a new state for a budget, we need the previous value and the amount of money transferred , 103

ordinal◁ (ordinaal, ordenbaar)
of quantities: the property that they can be ordered;
Example: numbers are ordinal , 65

orthogonal◁ (onafhankelijk, loodrecht)
for vectors or directions: being perpendicular to. In general: independent;
Example: in state charts: the behaviors of non-communicating systems, where every state in one system can occur in every state of the other system and vice versa , 95

orthogonal◁ (orthogonaal)
-of a taxonomy: a taxonomy where every property applies to all concepts;
Example: the periodic table of elements. Here every cell is a concept: a chemical element. Every element has a property group (column) and a period `period`

(row) , 136

parallel◁ (evenwijdig)
the property that a pair of two coplanar straight lines do not intersect;
Example: the velocity vectors in two points of a moving, but non-rotating rigid object are parallel , 20

parameter◁ (parameter)
a quantity, sometimes known and sometimes unknown, that occurs in a function or other expression;
Example: the parameter representation of a line between points p1 and p2 is p1 + lambda(p2-p1), lambda is a parameter , 62

partial derivative◁ (partiële afgeleide)
Informal: the partial derivative of the function $f(x, y)$ with respect to x is the ordinary derivative, while treating y as a constant;
(no example) , 43

periodic◁ (periodiek, regelmatig herhalend)
of a process: repeating itself after a constant time lapse;
Example: a pendulum, or a book-keeping system where each 1st of January the previous book-year is closed , 104

perpendicular◁ (loodrecht)
the maximal difference between two directions;
Example: North and East are perpendicular directions everywhere on the globe (except on the North pole and South pole) , 20

POset◁ (partieel geordende verzameling)
partially ordered set;
Example: the collection of intervals is partially ordered under the relation `greaterThan()` , 65

possible, logically◁ (logisch mogelijk)
of a concept: the values of its proporties do not contradict;
Example: Let us suppose that `big` and `small` are mutually exclusive values of the property `size`, and that `open` and `closed` are mutually exclusive values of the property `top`. Therefore, a big open box, a small closed box etc. are logically possible, whereas a big small box or a big open closed box are logically impossible , 63

postulate◁ (postuleren)
verb: formulate as a working hypothesis; substantive: a working hypothesis;
Example: the truth of the postulate can not be proven, but it is assumed true until evidently shown false. Unlike (normal) hypotheses, postulates are sometimes not subject to deliberate attempts of falsification. E.g., any formula to describe physical phenomena should be independent of the speed of the laboratorium where the phenomenon occurs , 25

prediction time◁ (voorspellingstijd)
the time period for which we want to, (or: are able to) obtain a valid prediction;
Example: for the weather, there are no known methods to obtain a prediction time longer than 5 or 6 days , 116

prediction, 1st kind◁ (voorspelling van de 1e soort)
unconditional prediction;
(no example) , 11

prediction, 2nd kind◁ (voorspelling van de 2e soort)
conditional prediction;
(no example) , 11

prediction, conditional◁ (voorwaardelijke voorspelling)
a statement about something that is going to happen provided that some condition is fulfilled, where this condition may or not may be under somebody's control;
Example: If I work hard enough, I will get I high grade for my exam on Modeling , 11

prediction, unconditional◁ (onvoorwaardelijke voorspelling)
a statement about something that is going to happen without the possibility to in-

fluence on the course of events;
Example: the weather forecast, as long as there is no technology to influence the weather, is an unconditional prediction. Also, under fair and legal circumstances, predictions of stock exchange rates are also unconditional , 11

prediction◁ (voorspelling)
- as a purpose of models: (1) a statement that at a given time point in the future something will happen, or (2) a statement that when something (not seen before) will happen, a certain property will be observed;
Example: (1) the next lunar eclipse will occur April 15, 2014 (this text is written December, 2012); (2) if we continue to consume fossile fuel in the current rate, sea levels will rise , 10

preposition◁ (voorzetsel)
word used to indicate mainly spatial or temporal relations;
Example: 'near', 'above', 'behind', 'before', 'during', ... 'Notwithstanding', although not referring to a spatial or temporal relation, is also a preposition , 56

presentation◁ (vertolking)
casting the result of formal operation with a model in a form that can be more easily understood in the context of the initial purpose;
Example: a model for predicting the weather produces a table with numerical values for the temperatures in a certain region as a function of time. These data are incomprehensible for most most stake holders; therefore they could be presented in the form of a map with small thermometers drawn in , 29

problem owner◁ (probleemeigenaar)
a person or group of people who benefit from the solution of the problem, or: who take the initiative for the problem being solved;
Example: for a model to predict the risk of aircraft failure, the problem owner could be the aircraft manufacturer, having primary interest in accurate estimates of this risk , 29

process◁ (proces, voortgang)
a behavior, or a collection of behaviors, of a dynamical system. The word 'process' in physics typically refers to one particular behavior; in computer science, it refers to a running program, which can display a variety of behaviors, perhaps depending on its input;
(no example) , 147
the development of some system over time;
Example: boiling an egg, doing a long division, performing a billiard shot , 17

process◁ (proces, voortgang)
the changes that a dynamical system undergoes when it develops over time, involving causes and effects;
Example: from physical processes (compression, expansion of gasses, propagation of waves or moving material objects) to social processes (the occurrence and resolution of conflicts, organization and reorganization of institutions , 88

projection◁ (projectie)
limiting the number of properties, or the number of values of properties of a system to reduce the state space;
Example: for a sock, in most cases we are not interested in exactly how dirty it is. The only values we want to distinguish are 'clean' or 'not clean'. , 93

properties, exposed -◁ (zichtbare eigenschappen)
properties in a conceptual model for which a value change in the modeled system is visible;
Example: in a clock with only an hour and a minute hand, the number of minutes in the present hour is exposed , 94

properties, hidden -◁ (onzichtbare eigenschappen)
properties in a conceptual model for which a value change in the modeled system

is not visible;
Example: in a clock with only an hour hand and a minute hand, the number of seconds since midnight is hidden , 94

property◁ (eigenschap)
a means for distinguishing concepts;
Example: `water` and `ice` are two very much related concepts. To distinguish them, we can use the property `aggregationPhase`. For water, the value of this property is `liquid`, for ice it is `solid` , 53
a pair (name, set of values);
Example: the property `height` of the concept `lantern` could be 6.0, stating the height of a lantern is the floating point number 6.0 (=a number of meters), that is: a set with only one element. It could e.g. also be the range of numbers between 5.0 and 8.0, stating that the lantern is anywhere between 5.0 and 8.0 meters high , 53
aspect of a concept that carries information;
Example: material things have properties such as `size`, `mass` and `aggregationPhase`. These are meaningless for a concept such as `pianoSonata`. Conversely, properties such as `duration` or `loudness`, applicable to `pianoSonata`, have no meaning for a concept such as `sandwich` or `briefcase` , 53

purpose◁ (doel, doelstelling)
-of a model: what the modeler wants from the model;
Example: optimization, decision support, verification are possible purposes of a model , 9

quantities, hidden◁ (verborgen grootheid)
non-exposed quantities: quantities that may or may not be present in a model, and that may help explain seemingly non-causal behavior in a system;
Example: the number of bacteria in the body of an infected, but not yet diagnosed patient during incubation time , 147

quantity◁ (grootheid)
a mathematical object that can assume a value;
Example: in a mechanical model, g, the gravity acceleration is an essential quantity , 61

random experiment◁ (door toeval bepaald, niet voorspelbaar)
A random experiment is an experiment that can result in different outcomes, even though it is repeated in the same manner every time;
Example: throwing of dice , 21

range◁ (bereik)
-of a function: the set of values a function can return;
Example: for the function 'square', the range consists of all non-negative reals , 56

range◁ (reeks, serie)
a set, where all elements of the set are known by knowing just a minimum and a maximum;
Example: the range of integers between 3 and 6 is the set with elements 3, 4, 5, 6 , 54

ranking◁ (ordenen in rang)
assigning integers to an ordered collection of items, such that the order of the integers matches with the order of the items;
Example: Olympic medal winners are ranked 1, 2 and 3 , 66

raw◁ (ruw, onbewerkt)
of data: data that results from observation or measurement, without any further processing;
Example: reading a thermometer, we see that the mercury level is halfway between the 12th and the 13th marks. '12.5' is a raw reading. Only by further processing, e.g., using the numeric labels near the marks, we can deduce that the temperature measured is, say,

21.5 centigrades , 27

reachable◁ (bereikbaar)
of a state S: if there is a path trough a state space, consisting of admitted transitions, leading from a reachable state to S, S is reachable;
Example: if a bulky sofa could be placed in an attic that has only a narrow spiral staircase as access, there must be a route for this sofa over the staircase such that it never gets stuck , 93

reason◁ (redeneren)
(verb): obtain the resulting value for a formal expression by applying rules of logic; opposed to calculate;
Example: 'all cars must have licence plates; a police car is a car, hence a police car must have a licence plate' is a valid reasoning. If both premises are true ('all cars must have licence plates', and 'police car is a car'), the conclusion ('a police car must have licence plate') is also true , 24

reconstruction◁ (reconstructie)
-in the context of sampling: the process to recover information about a continuous quantity from a set of sampled values of this quantity;
Example: the audible sound, produced from reading digital information from a CD , 18

recursion◁ (recursief)
the property that something is defined in terms of itself, or perhaps of earlier versions of itself;
Example: 'current age' could be defined as 'last year's age plus one year, or zero in the year you are born'. N factorial is defined as (N-1) factorial multiplied with N, where 0 factorial is defined as 1 , 105

reflection◁ (reflectie, bespiegeling)
the mental process of looking back to some achievement with the purpose to improve one's understanding and skills for future occasions;
Example: a modeler could conclude, that, despite her mathematical skills, she has difficulty in explaining the model outcomes in terms that make sense to the problem owner, and take corrective actions , 31

reflexive◁ (reflexief)
-of a relation: a reflexive relation applies between an item and itself;
Example: `hasSameFatherAs` , 146

regime◁ (regime, bereik)
- in models: a range of values for the quantities in a model such that a set of assumptions holds; or a range of values for the quantities in a model such that the behavior of the model is similar but different for another regime;
Example: consider a model for the physical properties of water. For temperatures and pressures in a certain range, water is solid; in another range it is liquid and in yet another range it is a gas. The behavior in one regime is similar over the entire regime (for instance, for a gas, volume and pressure are inversely proportional for a constant temperature. This is not true for liquid or solid.) , 32

relation, binary◁ (binaire relatie)
a relation with two terms;
Example: the relation `greaterThan()` is a binary relation , 129

relation, equivalence◁ (equivalentierelatie)
a relation that is reflexive, symmetric and transitive;
Example: `hasSameFatherAs` , 71

relation◁ (verband)
way to connect two or more concepts or their properties;
Example: `isMarriedTo()`, `greaterThan()`, but also formulas such as V=IR, relating V, I and R in Ohm's law , 56

representation◁ (vertegenwoordiging)
the relation between a concept in the model and its corresponding entity in the modeled system;
Example: p represents the pressure in

the vessel , 52
research◁ ((wetenschappelijk) onderzoek)
(as opposed to design) the systematic investigation of some object or phenomenon with the primary intention to gain understanding for the benefit of a scientific community;
Example: assessment of the value of a physical constant, checking of a hypothesis regarding the behavior of a system , 9
return value◁ (resultaatwaarde)
-of a function: the value, part of the range of the function, that is obtained by applying the function to its argument;
Example: for the function 'color', the return value is e.g. red, green or purple , 56

saddle point◁ (zadelpunt)
a point $(a, b, f(a, b))$ of $z = f(x, y)$ is called a saddle point if (a, b) is a critical point but $f(a, b)$ is not an extremum;
(no example) , 75
sample◁ (monster, steekproef)
(verb) represent the behavior of a large set by knowing relatively few values of that set;
Example: estimating the quality of a batch of oranges by testing the quality a handful (stochastic sampling), or storing a continuous signal (music) in the form of a large number of values, each 1/44100 second apart on a CD (digital audio sampling) , 106
sampling◁ (bemonsteren, een steekproef nemen)
representing a continuous quantity by a discrete one that comes in steps, small enough so that for practical purposes no information is lost;
Example: a motion pixture is a way to sample the visual impression of some scene by taking 24 snapshots per second , 18
sanity check◁ ((geen Nederlandse vertaling))
a simple test to see if a construct could make sense;
Example: looking at consistency, order of magnitude, or dimensional correctness can reveal if a formula or a numerical outcome could make sense , 37
scalar◁ (scalair getal)
a real number, in contrast with a vector or a matrix;
Example: a vector can be multiplied with a scalar to yield another vector , 149
scale, interval◁ (intervalschaal)
scale that allows addition and subtraction;
Example: temperature scale in Centigrade , 67
scale, Mohs◁ (schaal van Mohs)
example of an ordinal scale that is not an interval scale: if for two minerals, A and B, A receives scratches and B does not when rubbed against each other, then B's Mohs number is higher than that of A;
Example: diamond is harder than steel and steel is harder than chalk , 66
scale, ordinal◁ (ordinale schaal)
a scale that allows ordering and median computation;
Example: Mohs scale , 66
scale, ratio◁ (ratioschaal)
a scale that allows the calculation of ratio's;
Example: the Kelvin scale and many other scales for physical quantities , 67
scope◁ (toepassingsgebied)
-of a model: range of modeled situations to which a model should apply, or for which a model should be useful;
Example: water could be modeled as an ideal gas, provided that the temperature is not too low and/or the pressure is not too high , 35
segmentation◁ (opdeling)
separating a unity into meaningful segments;
Example: a city map is segmented into street blocks; a year schedule may be segment in semesters, trimesters or quartiles, and an living organism may be segmented into digestive, reproductive, respirational and other functional seg-

ments , 51
semantic network◁ (semantisch netwerk)
entity relation graph;
(no example) , 61
servicing◁ (afhandelen)
- of an external event: the act of responding to that event from within a process;
Example: somebody is cooking, and the phone rings; in responding to the phone, the stove has to be put off first , 98
simplex method◁ (simplex methode)
a method to solve optimization problems with inequality constraints that are linear expressions of the unknowns;
(no example) , 147
simulation◁ (nabootsing)
for some proces P1, a simulation P2 is a second process that aims to replicate certain aspects of P1;
Example: the game of chess simulates traditional warfare , 142
simulation◁ (naspelen)
the evaluation of subsequent states of a dynamic model by unrolling the recursive definition of quantities in state i+1 in terms of their values in state i, beginning at some starting state i=0;
Example: using the bank transcripts over a period of time to reconstruct and analyse the time-behavior of a bank account , 105
singleton◁ (singleton)
a set that contains only one element;
Example: the set of monuments in Paris being taller than 300 meters , 54
solution curve◁ (oplossingskromme)
one of the solutions y=y(x) of a differential equation, relating y' to a function of y and possibly x;
Example: a solution curve for y'=f(x,y)=y is the function y=3 exp(x), another one is y=0. , 113
solution, closed form◁ (oplossing in gesloten vorm)
the solution of a problem, given in terms of a finite set of arithmetical operations, so that generic properties of the solution can be stated without having to rely on numerical estimates;
Example: the solution for x of the equation ax+b = y for any a, b, y, a different from 0, is x=(b-y)/a , 10
spatial◁ (ruimtelijk, ruimteachtig)
regarding space; see also temporal;
Example: length, width, and height are spatial dimensions, as well as 'per length' (spatial frequency, as in 'this necklace has 3 beads per centimeter') , 92
specification◁ (specificatie)
- as a purpose for models: make sure that something in the modeled system will occur of will be realized, or give a description of some artefact that is sufficiently complete so that the artefact can be realized (purely) on the basis of the specification, perhaps still allowing for open choices;
Example: a blue print of a piece of furniture (say, a chair) is a model of that chair - made at at a time where the chair doesn't yet exist, but (perhaps) leading to the actual existence of the chair once it is build , 13
stable solution◁ (stabiele oplossing)
a solution (e.g., of a differential equation) such that, if (initial) conditions are different, the solution is hardly or not different;
Example: For a differential equation, 'stable' means that a solution close to a so-called equilibrium, will approach that equilibrium , 117
stake holder◁ (betrokkene)
a person or group of people who are, purposely or involuntary, affected by the solution of some problem or the failure thereof;
Example: an insurance company is a stakeholder for a model to predict the effectiveness of a pharmaceutic drug , 29
state chart◁ (toestandsdiagram)
a process model where a process is a graph, the nodes being states and the edges

transitions between states;
Example: an illustrated manual, showing step-by-step the assembly of a piece of furniture , 89

state space explosion◁ (explosie van de toestandsruimte)
the phenomenon that the number of states, and the number of possible processes, of a system, grow exponentially, both in the number of properties of the system, and the number of steps of the process;
Example: the game of chess continues to fascinate players because its state space is intractably large , 92

state space◁ (toestandsruimte)
the collection of all possible states of a system, that is: all possible bindings of the properties of the system to values;
Example: all possible configurations of 54 colored squares that can be obtained by rotating any of the six faces of Rubick's cube form the state space of this cube , 91

state transition◁ (toestandsovergang)
the replacement of one binding of values to the properties in a system to another binding;
Example: somebody celebrating his birthday, where the property age, originally bound to N, now gets bound to N+1 , 89

state, initial◁ (begintoestand)
in a dynamic process: the state that is occupied by the process, when no transitions have yet taken place;
Example: the initial state of all symphonies is a conductor raising his baton, the orchestra being silent , 93

state◁ (toestand)
the snapshot of a system, containing all its quantities and their current values;
Example: Rembrandt's Night watch depicts one state in the history of the 17th Century Amsterdam police force , 89

static◁ (statisch, stationair, tijdsonafhankelijk)
not involving time; opposite to dynamic;
Example: the balance of forces that keep a building from collapsing under its own weight , 17

stationary point◁ (stationair punt)
an argument value x of a function f(x) such that sufficiently small change of x causes the value of f to stay the same;
Example: a local maximum or minimum of a function , 19

stationary◁ (stationair, onveranderlijk)
not varying as a function of time or another argument, constant;
Example: the value of physical constants such as the speed of light and the mass of a proton , 17

steer◁ (besturen)
- as a purpose of models: based on a representation of X, and measurements on some aspects of X or its environment, perform actions that influence X;
Example: a thermostat influences the working of a heater, thereby influencing the temperature in a room , 14

step size◁ (stapgrootte)
-in sampling: the increment of a sampled quantity;
Example: in film, 1/24th of a second; in CD's: 1/44100th of a second , 142

stochastic◁ (stochastisch, door toeval bepaald)
involving randomness;
Example: the motion of molecules in a gas , 21

straight◁ (recht)
the property of a curve passing through two given points, that its length is minimal;
Example: the trajectory of a falling point mass in vacuum with zero initial speed is a straight line , 20

structure, impose◁ (structuur opleggen)
define meaningfully chosen aggregating concepts to aggregate related concepts so that a simpler conceptual model results;
Example: classifying all materials into the concepts `liquid`, `solid` and `gaseous` , 136

subjective◁ (subjectief)
depending on an individual observer, or on

an individual opinion. Opposite of objective;
Example: John likes marshmallows , 11

subscript notation◁ (subscriptnotatie)
notation to address a property of a concept: for concept C, P with subscript C is the value of property P;
(no example) , 133

symbolic◁ (symbolisch)
- of a modeling strategy: performing operations upon mathematical or logical expressions, not dealing with values only. Opposite to numeric;
Example: a+b = b+a is a symbolic expression, denoting the property that addition is commutative, as in 3+4= 4+3 or 17+1=1+17 , 19

symmetric◁ (symmetrisch)
-of a relation: a symmetric relation between A and B also holds between B and A;
Example: `marriedTo` , 146

symmetry◁ (symmetrie)
the condition that only part of a system needs to be known in order to know something about the entire system;
Example: if only the left hand part of the shape of a mirror-symmetric piece of clothing is drawn, a capable tailor can make the entire piece. Symmetry can be spatial, but temporal and other symmetries occur as well , 92

synchronous◁ (gerelateerd in tijd, in de maat)
of an event with respect to some process P: occurring between two predefined state transitions in P. Opposite to asynchronous;
Example: when preparing a sandwich, the butter should be applied in between slicing the bread and putting on the topping. Applying butter is to be synchronized with the other two stages of the process , 96

tangent plane◁ (raakvlak)
the tangent plane to the graph $z = f(x, y)$ at (a, b) is given by $f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$;
(no example) , 45

taxonomy◁ (taxonomie)
an ontology, typically used to systematically list properties of entities;
Example: the Linnaeus categorization system of living beings , 136

temporal◁ (tijdachtig)
regarding time; see also spatial;
Example: duration and (temporal) frequency are temporal dimensions , 92

term◁ (term)
-of a relation: an argument expression the relation says something about;
Example: the relation $>$ has two terms (arguments); the relation holds if the first argument is larger than the second. The arity of a relation is the number of terms it takes as arguments , 129
a quantity, often occurring in a sum expression. See also factor;
Example: in (a+b)(a-b)=a x a - b x b, a, b, a x a and b x b are terms whereas (a+b) and (a-b) are factors. In a x a are two factors a , 62

theory◁ (theorie)
A body of knowledge, accepted by a community, used to explain empirical phenomena;
Example: the theory of electromagnetism in physics, the theory of consumers' behavior in economy , 10

thought experiment◁ (gedachte-experiment)
attempt to find some contingent fact by mere reasoning, without any empirical observation;
Example: Galilei was interested in the speed of falling objects: would speed depend on weight? Prior to the famous leaning tower experiment with the two unequal canon balls, he postulated the opinion that both would fall with equal speeds, based on a thought experiment: suppose that a canon ball would be cut in two equal halves, where the two stay in close contact during falling, would this influence the falling behavior? , 142

time lapse◁ (verstreken tijd)

the amount of time between two subsequent transitions in a model with full time ordering;
Example: a day is the time lapse between rising and setting sun , 100

time reversibility◁ ((tijd)omkeerbaarheid)
the aspect of time in physical processes that, at microscopic scale, past and future can be interchanged;
Example: a hypothetical film of a microscopic process can be viewed in reverse without showing any non-physical behavior , 87

trace◁ (spoor)
a route through a state space;
Example: the footprints left by an animal walking through the wood represent a trace in its state space , 91

train◁ (trainen, instrueren)
- as a purpose of models: assist trainees to get familiar with some system X, avoiding the risks if untrained personnel would work with the actual X;
Example: flight simulators to train pilots; simulators of industrial plants to train operators; anatomical simulators to train surgeons , 14

transition, internal◁ (interne overgang)
transition that is not caused by an external event;
Example: when a balloon is gradually inflated, there is a moment where it explodes. This happens when the stress in the balloon exceeds the strength of its skin. It does not require any external trigger , 98

transition◁ (overgang)
short for state transition;
(no example) , 89

transitive◁ (transitief, overdraagbaar)
the property of a relation R(A,B), implying: if R(A,B) and R(B,C) then R(A,C);
Example: `greaterThan` , 65

tree◁ (boom)
special kind of graph: every node except one (called the 'root') has exactly one other node, called its parent. A parent can have multiple children;
Example: a hierarchy or a family tree, restricted to male (or female, but not both) descendants , 136

type, compound◁ (samengesteld type)
an element of a compound type is a concept that has one or more properties that themselves have a type;
Example: the type `circle` has properties `radius` and `midpoint`. These have a type: `radius` is a real number, and `midpoint` has type `point` , 62

type, elementary◁ (elementair type)
types that are not compound, a.k.a. atomic types;
Example: the type Boolean is the set with elements `TRUE,FALSE`. The values `TRUE` and `FALSE` are not compound , 62

type◁ (type, soort, specimen, verzameling van waarden)
-of a property is the set of values that can be associated to that property;
Example: a property with type `integer` can take any integer value , 53
-of a quantity: the set of values that can be assumed by that quantity;
Example: the set with elements `red, green, blue, ...` is the type of a quantity named `color` , 61

unification◁ (unificatie)
- as a purpose of models: providing a representation that allows explanations and perhaps even predictions in two or more domains that initially were thought to be unrelated;
Example: the flow of gas through a pipe and the flow of traffic on a motor way can, under some circumstances, be unified , 11

unstable◁ (instabiel)
the behavior of numerically evaluated recursive simulations that, typically due to too coarse sampling, approximated solutions get further and further off;
Example: a mass spring system, approximated by discrete sampling, where

the sampling time step is not small compared to the period, may get instable , 151

value◁ (waarde)
- in design: that which is to be produced in a design process. Examples of value could be: profit, safety, health, amusement, ...;
Example: in designing a vehicle, values could be safety, speed, and comfort , 12
-of a property: in case the set of values of a property is limited to a singleton, the (single) element of this singleton is called 'the value of the property';
Example: '40000 km' is the value of the property 'perimeter' of the concept 'earth' , 54

variable◁ (variabele)
a quantity that can take several values, or an unknown in an equation;
Example: most text would not call pi = 3.1415... a variable, whereas in the equation 2x = pi, x would be called a variable , 62

variable◁ (variabele)
quantity that occurs as argument of a function;
Example: In y=f(x), x is called a variable. If f is a non-trivial function, y will assume different values as well (if not, the notation y=f(x) is misleading); therefore it is common to call y also a variable. Sometimes x is called an independent variable, whereas y is called dependent variable, as it depends on x , 41

verification◁ (verificatie, zekerstelling)
- as a purpose of models: assessing if something in the modeled system is true;
Example: in a control system for railway signaling, the software should be such that it is impossible that two trains are allowed in the same block. To verify if this property holds, a model of the software is constructed. Verification is a particular instance of analysis , 13

white box◁ (witte doos)
other term for glass box;
(no example) , 25